Orientation on Mathematical Research

The Yoneda Lemma

November 2023

Supervisor: Dr.Sara Mehidi

Authors: Q. van der Velden (6872018)
B. Kiezebrink (5257956)
J. Grube (2850222)
W. Price (0972657)
Rashiqa Dawood (1916890)

Department of Mathematics
Utrecht University

Table of Contents

1 Introduction

When one looks at subjects in mathematics one may notice that, indeed, subjects like the theory of groups or the theory of topological spaces are built up in a similar manner. In either case, we consider certain objects which are groups and topological spaces respectively together with relations between these objects, these are group homomorphisms and continuous functions respectively. Naturally, we may then wonder whether there are mathematical subjects in which these objects and relations follow similar rules. This brings us to category theory.

Category theory, in a way, is to mathematics what abstract algebra is to topics such as geometry and number theory. It in fact gives us a way to describe mathematical structures such as the theory of groups or topological spaces in a more general sense, allowing us to relate these subjects themselves to each other.

The goal in this report will be to understand and discuss applications of a central theorem in category theory, typically referred to as the "Yoneda Lemma", named after the Japanese mathematician Nobuo Yoneda. The Yoneda Lemma tells us that the objects of any category (the category of sets, groups, topological spaces, rings etc), can be determined "up-to-isomorphism" by looking at the structure of morphisms (e.g. functions, group homomorphisms, continuous maps etc) into, or out of them. Furthermore, it gives us a way of embedding any category $\mathcal{C}$ inside a larger "presheaf" category $\widehat{\mathcal{C}}$, and the Yoneda Lemma enables us to perform constructions in this category via $\mathcal{C}$.

In order to get to understanding the Yoneda Lemma itself however, we must first discuss the fundamental theory of category theory. We first prove a basic form of the Yoneda Lemma, accompanied by some simple examples to help build an intuition for the correspondence. Then we build the necessary structure to state and prove a stronger form of the Yoneda Lemma, which we refer to as the Yoneda isomorphism. The final sections of the report explore some ways of applying the Yoneda Lemma to various areas of mathematics.

2 Basics of Category Theory

2.1 Categories

In this section we introduce the basic concepts of category theory that are needed to understand the Yoneda Lemma. The primary content of this report begins with the sections on Representable Functors and the Yoneda Lemma, so the reader is advised to read ahead if possible, and use this section as a reference for concepts they are not familiar with. The recommended reference for more category theory background is *Category Theory in Context* (Reihl. [1]), where many examples of the concepts discussed in this section can be found.
We first introduce the concept of a category.

Definition 2.1 (Category)**.** A category $\mathcal{C}$ is a pair consisting of a collection $\mathrm{Obj}(\mathcal{C})$ of *objects* and a collection $\mathrm{Arr}(\mathcal{C})$ of *arrows* (also called *morphisms*). The elements from $\mathrm{Obj}(\mathcal{C})$ are called $\mathcal{C}$-objects and the elements from $\mathrm{Arr}(\mathcal{C})$ are called $\mathcal{C}$-arrows. On $\mathcal{C}$ we have the following structure:

1. For each $\mathcal{C}$-arrow f there are distinguished objects $\mathrm{dom}(f), \mathrm{cod}(f) \in \mathrm{Obj}(\mathcal{C})$ (the domain and codomain of f). We write $f : A \to B$ to say that $\mathrm{dom}(f) = A$ and $\mathrm{cod}(f) = B$.

2. For two $\mathcal{C}$-arrows f, g with $\mathrm{cod}(f) = \mathrm{dom}(g)$ there is another arrow, the *composition* of f and g denoted $g \circ f$ with $g \circ f : \mathrm{dom}(f) \to \mathrm{cod}(g)$.

3. For three $\mathcal{C}$-arrows f, g, h with $f : A \to B, g : B \to C, h : C \to D$ we have $h \circ (g \circ f) = (h \circ g) \circ f$. We say that composition is *associative*.

4. For a $\mathcal{C}$-object A there is a distinguished *identity* arrow $\mathrm{Id}_A : A \to A$, such that for all $\mathcal{C}$-arrows $f : A \to B, g : C \to A$, $f \circ \mathrm{Id}_A = f$ and $\mathrm{Id}_A \circ g = g$. We also use the notation 1_A for this arrow.

Definition 2.2. In a category $\mathcal{C}$, fixing a pair of objects A, B defines a class $\mathrm{Hom}(A, B) := \{f \in \mathrm{Arr}(\mathcal{C}) \mid \mathrm{dom}(f) = A \wedge \mathrm{cod}(f) = B\}$, and these classes form a partition of $\mathrm{Arr}(\mathcal{C})$. To specify the category in the notation, we write $\mathrm{Hom}_{\mathcal{C}}(A, B)$ or $\mathcal{C}(A, B)$. When this class forms a *set*, it is called a *hom-set*, and a category $\mathcal{C}$ in which $\mathrm{Hom}(A, B)$ is a set for every pair of objects $A, B \in \mathrm{Obj}(\mathcal{C})$, is called a *locally-small category*. A *small category* is a locally-small category in which $\mathrm{Obj}(\mathcal{C})$ is also a set. A *large category* is a category which is not small.

Remark 2.3. The categories of interest in this report are all *locally-small*, because the Yoneda Lemma is about locally-small categories. Instead of defining $\mathrm{Arr}, \mathrm{dom}, \mathrm{cod}$, it is natural to define a locally-small category $\mathcal{C}$ by directly defining the hom-sets $\mathcal{C}(A, B)$ for every pair of objects A, B. Identities are given as distinguished elements $\mathrm{Id}_A \in \mathcal{C}(A, A)$, and the composition operation can be described as a family of object-parametrised functions $\circ_{A,B,C}$ for $A, B, C \in \mathrm{Obj}(\mathcal{C})$ (functions - because these are sets):

$$\begin{aligned} \circ_{A,B,C} : \mathcal{C}(B, C) \times \mathcal{C}(A, B) &\longrightarrow \mathcal{C}(A, C) \\ (g, f) &\longmapsto g \circ f \end{aligned}$$

The class $\mathrm{Arr}(\mathcal{C})$ can be recovered as $\cup_{A,B\in\mathrm{Obj}(\mathcal{C})}\mathrm{Hom}_{\mathcal{C}}(A,B)$, and $\mathrm{dom}(f),\mathrm{cod}(f)$ refer to the objects A,B such that $f\in\mathrm{Hom}_{\mathcal{C}}(A,B)$. Whenever f,g are composable, i.e. $\mathrm{cod}(f)=\mathrm{dom}(g)$, the arrow $g\circ f$ is $g\circ_{A,B,C} f$, where $A:=\mathrm{dom}(f), B:=\mathrm{cod}(f)=\mathrm{dom}(g), C:=\mathrm{cod}(g)$. Typically the definition of $\circ_{A,B,C}$ is uniform in the parameters, so we just describe $\circ$, understanding that it must restrict appropriately to the relevant hom-sets.

We now introduce a few examples of categories. It is readily checked that these are indeed categories

Example 2.4.

1. The category **Set** of sets has the class of sets as objects and for sets A,B, $\mathbf{Set}(A,B)$ is the set of functions from A to B. Composition of functions is the standard composition, i.e. $(g\circ f)(x)=g(f(x))$, and Id_A is the identity function $\mathrm{Id}_A(x)=x$.

2. The category **Grp** of groups has the class of groups as objects and, for groups G,H, $\mathbf{Grp}(G,H)$ is the set of group homomorphisms from G to H. Composition and identities are the same as in **Set** (one can check these always yield group homomorphisms).

3. Similarly as for groups we have the category **Ab** of abelian groups and group homomorphisms, the category **Ring** of rings (with identity) and ring homomorphisms, the category **Cring** of commutative rings and ring homomorphisms, the category **Mon** of monoids and monoid homomorphisms and the category **Cmon** of commutative monoids and monoid homomorphisms.

4. An example of a category with non function-like morphisms is given by taking the category with as objects all elements of $\mathbb{Z}$ and as arrows the total order $\leq$ on $\mathbb{Z}$, such that for $a,b\in\mathbb{Z}$ we have a unique arrow $a\to b$ precisely when we have $a\leq b$. It is then possible to check that indeed all axioms of a category hold.

5. Consider a topological space $\mathcal{T}$. Take the objects of our category to be the open sets in $\mathcal{T}$ and let our arrows be the inclusion $\subseteq$ of these sets. For two open sets U and V in $\mathcal{T}$ we then have an arrow $U\to V$ whenever $U\subseteq V$.

2.2 Morphisms in a category

We prove various lemmas about morphisms in a general category $\mathcal{C}$.
In the categories of Example 2.4, the identity arrows are always the unique arrow that behaves as a left and right identity with respect to composition (i.e. the condition on Id_A). This is easily proved for a general category.

Lemma 2.5. Let $I:A\to A$ be an arrow such that for all $f:A\to B, g:C\to A$, $f\circ I=f$ and $I\circ g=g$. Then $I=\mathrm{Id}_A$.

Proof. $I=I\circ\mathrm{Id}_A=\mathrm{Id}_A$. The first equality uses $f\circ\mathrm{Id}_A=f$ for any f with $\mathrm{dom}(f)=A$, and the second uses the assumption $I\circ g=g$ for any g with $\mathrm{cod}(g)=A$. $\square$

Definition 2.6. Let $\mathcal{C}$ be a category and $f : A \to B$ a $\mathcal{C}$-arrow.

1. A *right inverse* of f is a morphism $g : B \to A$ such that $g \circ f = \mathrm{Id}_A$
2. A *left inverse* of f is a morphism $h : B \to A$ such that $f \circ h = \mathrm{Id}_B$
3. f is an *isomorphism* if has both a left and right inverse.

Right (resp. left) inverses to f are not necessarily unique, but if f is an isomorphism, then there is only one left inverse and only one right inverse, and they are the same morphism. When it exists, we refer to it as "the inverse of f", written f^{-1}.

Lemma 2.7. Let $f : A \to B$ be an isomorphism, with a right inverse $g : B \to A$ and a left inverse $h : B \to A$. Then $g = h$, and moreover, g is both the unique right inverse and the unique left inverse of f.

Proof. $g = g \circ \mathrm{Id}_B = g \circ f \circ h = \mathrm{Id}_A \circ h = h$, and if $g', h' : B \to A$ are any another right (resp. left for h') inverse of f, then $g' = g' \circ \mathrm{Id}_B = g' \circ f \circ h = \mathrm{Id}_A \circ h = h = g$, and $h' = \mathrm{Id}_A \circ h' = g \circ f \circ h' = g \circ Id_B = g$. $\square$

Aside from inverses we wish to define terms for when a left or right cancellation law holds for morphisms.

Definition 2.8. Let $f : A \to B$, $g : B \to C$ and $h : B \to C$ be morphisms. If $g = h$ whenever $g \circ f = h \circ f$ we say that f is an *epimorphism*. If $g = h$ whenever $f \circ g = f \circ h$ we say that f is a *monomorphism*.

2.3 Functors

In Example 2.4, we saw that the collection of objects of a certain mathematical structure (groups, rings) forms a category where the morphisms are the appropriate notion of structure-preserving map between two such structures. This idea applies to the mathematical structure of a category itself, and so we define the relevant class of structure-preserving morphisms for between categories, called a *functor*.

Definition 2.9. For categories $\mathcal{C}, \mathcal{D}$, a *functor* $F : \mathcal{C} \to \mathcal{D}$ is a pair of assignments $F_0 : \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$ and $F_1 : \mathrm{Arr}(\mathcal{C}) \to \mathrm{Arr}(\mathcal{D})$ such that for every object A in $\mathcal{C}$ we have $F_1(\mathrm{Id}_A) = \mathrm{Id}_{F_0(A)}$ and for every $\mathcal{C}$-arrow $f : A \to B$ we have $F_1(f) : F_0(A) \to F_0(B)$. Finally for $\mathcal{C}$-arrows $A \xrightarrow{f} B \xrightarrow{g}$ we have $F_1(g \circ f) = F_1(g) \circ F_1(f)$. We often write $F_0(A) = FA$ and $F_1(f) = F(f)$ (with or without parentheses). We call these conditions on the F_0, F_1 the *functoriality axioms*, which say that functors preserve identities and compositions.

Remark 2.10. For locally-small categories $\mathcal{C}, \mathcal{D}$, specifying F_1 amounts to supplying a family of functions $\{\mathcal{C}(A, B) \to \mathcal{D}(FA, FB)\}_{A,B}$

Definition 2.11. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between locally-small categories. F is called *full* if the function

$$\begin{aligned} \mathcal{C}(A,B) &\longrightarrow \mathcal{D}(FA,FB) \\ f &\longmapsto Ff \end{aligned}$$

is surjective for all $A, B \in \mathrm{Obj}(\mathcal{C})$ and *faithful* if it is injective for all $A, B \in \mathrm{Obj}(\mathcal{C})$. We call a functor *fully-faithful* if it is full and faithful.

Lemma 2.12. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then F preserves isomorphisms.

Proof. Let $f : A \to B$ be an isomorphism in $\mathcal{C}$ with inverse $g : B \to A$. Then $\mathrm{Id}_{F(B)} = F(\mathrm{Id}_B) = F(f \circ g) = F(f) \circ F(g)$, and $\mathrm{Id}_{F(A)} = F(\mathrm{Id}_A) = F(g \circ f) = F(g) \circ F(f)$. Thus, $F(f)$ is an isomorphism in $\mathcal{D}$ with inverse $F(g)$. □

Definition 2.13. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. We say that F *reflects* isomorphisms,if for any morphism $f : C \to D$, if $F(f) : F(C) \to F(D)$ is an isomorphism then so is $f : C \to D$.

Lemma 2.14. Let $F : \mathcal{C} \to \mathcal{D}$ be a fully-faithful functor. Then F reflects isomorphisms.

Proof. Let $f : C \to D$ such that $F(f) : F(C) \to F(D)$ is an isomorphism. Let $h : F(D) \to F(C)$ be the inverse of $F(f)$. Since F is full there exists $g : D \to C$ such that $F(g) = h$. And so $F(\mathrm{Id}_D) = \mathrm{Id}_{F(D)} = F(f) \circ h = F(f) \circ F(g) = F(f \circ g)$, and because F is faithful we get $f \circ g = \mathrm{Id}_D$. We also have $F(\mathrm{Id}_C) = F(g \circ f)$ and again by faithfulness of the functor F, we get $g \circ f = \mathrm{Id}_C$. □

Definition 2.15. A functor $F : \mathcal{C} \to \mathcal{D}$ is an embedding if it is full, faithful and injective on objects.

Definition 2.16. Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be categories, and $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{E}$ be functors. We define their *composition*, $G \circ F$, to be the pair consisting of $(G \circ F)_0 = G_0 \circ F_0 : \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{E})$ and $(G \circ F)_1 = G_1 \circ F_1 : \mathrm{Arr}(\mathcal{C}) \to \mathrm{Arr}(\mathcal{E})$. We verify this is functorial (that it preserves identities and compositions). For a $\mathcal{C}$-object A we have

$$(G \circ F)_1(\mathrm{Id}_A) = G_0(F_0(\mathrm{Id}_A)) = G_0(\mathrm{Id}_{F_0(A)}) = \mathrm{Id}_{G_0(F_0(A))} = \mathrm{Id}_{(G \circ F)_0(A)}$$

Let now $f : A \to B$ be a $\mathcal{C}$-arrow. Then $F_1(f) : F_0(A) \to F_0(B)$ and consequently $G_1(F_1(f)) : G_0(F_0(A)) \to G_0(F_0(B))$, or equivalently $(G \circ F)_1(f) : (G \circ F)_0(A) \to (G \circ F)_0(B)$. Finally for $\mathcal{C}$-arrows $A \xrightarrow{f} B \xrightarrow{g} C$ we find

$$\begin{aligned} (G \circ F)_1(g \circ f) &= G_1(F_1(g \circ f)) \\ &= G_1(F_1(g) \circ F_1(f)) \\ &= G_1(F_1(g)) \circ G_1(F_1(f)) \\ &= (G \circ F)_1(g) \circ (G \circ F)_1(f). \end{aligned}$$

We conclude that $G \circ F$ is a well-defined functor $\mathcal{C} \to \mathcal{E}$.

Definition 2.17. Let $\mathcal{C}$ be a category. The *identity functor* $\mathrm{Id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is defined on objects by $(\mathrm{Id}_{\mathcal{C}})_0(A) = A$ and on arrows by $(\mathrm{Id}_{\mathcal{C}})_1(f) = f$. We verify functoriality. For $A \in \mathrm{Obj}(\mathcal{C})$, $(\mathrm{Id}_{\mathcal{C}})_1(\mathrm{Id}_A) = \mathrm{Id}_A = \mathrm{Id}_{(\mathrm{Id}_{\mathcal{C}})_0(A)}$, and if $f : A \to B$ is a $\mathcal{C}$-arrow, then $(\mathrm{Id}_{\mathcal{C}})_1(f) = f : A \to B$, or equivalently $(\mathrm{Id}_{\mathcal{C}})_1(f) : (\mathrm{Id}_{\mathcal{C}})_0(A) \to (\mathrm{Id}_{\mathcal{C}})_0(B)$. Let finally $A \xrightarrow{f} B \xrightarrow{g} C$ be $\mathcal{C}$-arrows. Then we have

$$(\mathrm{Id}_{\mathcal{C}})_1(g \circ f) = g \circ f = (\mathrm{Id}_{\mathcal{C}})_1(f) \circ (\mathrm{Id}_{\mathcal{C}})_1(g).$$

Lemma 2.18. For a category $\mathcal{C}$, $\mathrm{Id}_\mathcal{C}$ is the identity with respect to composition of functors.

Proof. Let $F : \mathcal{D} \to \mathcal{C}$ be a functor. Then for $D \in \mathrm{Obj}(\mathcal{D})$, $(\mathrm{Id}_\mathcal{D} \circ F)_0(D) = (\mathrm{Id}_\mathcal{D})_0(F_0(D)) = F_0(D)$ and for $f : D \to D'$

$$(\mathrm{Id}_\mathcal{D} \circ F)_1(f) = (\mathrm{Id}_\mathcal{D})_1(F_1(f)) = F_1(f)$$

So $\mathrm{Id}_\mathcal{D} \circ F = F$, since their assignments on objects and arrows agree. A similar proof shows $G \circ \mathrm{Id}_\mathcal{D} = G$. $\square$

Lemma 2.19. Composition of functors is associative.

Proof. Suppose we have a chain $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E} \xrightarrow{H} \mathcal{S}$ of composable functors between categories.

$$(H \circ (G \circ F))_0 = H_0 \circ (G \circ F)_0 = H_0 \circ (G_0 \circ F_0) = (H_0 \circ G_0) \circ F_0 = (H \circ G)_0 \circ F_0 = ((H \circ G) \circ F)_0$$

and similarly $(H \circ (G \circ F))_1 = ((H \circ G) \circ F)_1$. We conclude that $H \circ (G \circ F) = (H \circ G) \circ F$. $\square$

Corollary 2.20. The class of small categories and functors between them forms a large category **Cat**, with composition and identities for functors as above.

Proof. For a functor $F : \mathcal{C} \to \mathcal{D}$ we define $\mathcal{C}$ to be the domain of F and $\mathcal{D}$ to be the codomain (note this makes composition well-defined). The previous lemmas verify the identity and associativity axioms. $\square$

Remark 2.21. The restriction to small categories is necessary, since the class of all small-categories is a proper class. Larger categories of categories can be formed, but to make these considerations precise, one must choose their foundations. See [2] for further details.

2.4 Natural Transformations

A functor $F : \mathcal{I} \to \mathcal{D}$ can be thought of as a family of objects in $\mathcal{D}$ indexed by the category $\mathcal{I}$. For each $i \in \mathrm{Obj}(\mathcal{I})$, we have an object $D_i := F(i) \in \mathrm{Obj}(\mathcal{D})$, and morphisms $i \to j$ in $\mathcal{I}$ become morphisms between D_i and D_j (via F).

$$\begin{tikzcd}
& \mathcal{I} & & & & \mathcal{D} & \\
1 \arrow[rr, bend left, "a"] \arrow[dr, bend left, "c"] & & 2 \arrow[dl, bend left, "b"] & \xrightarrow{F} & D_1 \arrow[rr, bend left, "Fa"] \arrow[dr, bend left, "Fc"] & & D_2 \arrow[dl, bend left, "Fb"] \\
& 3 \arrow[ul, bend left, "e"] & & & & D_3 \arrow[ul, bend left, "Fe"] & \\
4 \arrow[uu, bend left, "d"] & & & & D_4 \arrow[uu, bend left, "Fd"] & &
\end{tikzcd}$$

This structured collection of objects and morphisms is called a *diagram in $\mathcal{D}$ of shape $\mathcal{I}$*, and is an equivalent perspective on what a functor is. Instead of starting with the category $\mathcal{I}$, one can start with a choice of objects and morphisms in $\mathcal{D}$, and derive the appropriate "shape category" $\mathcal{I}$ that makes those choices into a functor.
Given two functors/diagrams $F, G : \mathcal{I} \to \mathcal{D}$, we might compare their diagrams by defining pointwise morphisms $F(i) \to G(i)$ in $\mathcal{D}$. This gives us a notion of "2-morphism" between morphisms in **Cat** (i.e. functors), and for this notion to be well-behaved, we need this pointwise morphisms to be "compatible" with the morphisms within each diagram. Such a family of morphisms is called a *natural transformation*, which we define below.

Definition 2.22 (Natural transformation)**.** Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. Then a collection of arrows $\eta = (\eta_C : F(C) \to G(C) | C \in \mathrm{Obj}(\mathcal{C}))$ is called a *natural transformation* from F to G, written $\eta : F \Rightarrow G$ if for each $\mathcal{C}$-arrow $f : C \to C'$ the following diagram commutes:

$$\begin{tikzcd}
F(C) \arrow[r, "\eta_C"] \arrow[d, "F(f)"'] & G(C) \arrow[d, "G(f)"] \\
F(C') \arrow[r, "\eta_{C'}"'] & G(C')
\end{tikzcd}$$

2.5 New categories from old

If we have some category $\mathcal{C}$ we can construct from it a category $\mathcal{C}^{\mathrm{op}}$: the *opposite category* of $\mathcal{C}$. This category is as follows:

- Objects are precisely the objects of $\mathcal{C}$.

- Arrows are precisely the arrows of $\mathcal{C}$ with their domain and codomain interchanged. That is, if in $\mathcal{C}$ we have that f is an arrow $C \to C'$ then in $\mathcal{C}^{\mathrm{op}}$ we have that f is an arrow $C' \to C$. We often write $\overline{f}$ for some arrow $f : C \to C'$ in $\mathcal{C}$ viewed as an arrow $C' \to C$ in $\mathcal{C}^{\mathrm{op}}$.

- Composition is given by composition in $\mathcal{C}$. That is, for $C \xrightarrow{\overline{f}} C' \xrightarrow{\overline{g}} C''$ the composition is given by $\overline{g} \circ \overline{f} = \overline{f \circ g}$.

- Identities are precisely the identities from $\mathcal{C}$. That is from some object C from $\mathcal{C}^{\mathrm{op}}$ the identity in $\mathcal{C}^{\mathrm{op}}$ at C is just the identity at C in $\mathcal{C}$.

It is readily shown that the above indeed gives us a category. The opposite category can be seen as the result of flipping all the arrows in a category. We can now also consider functors from $\mathcal{C}^{\mathrm{op}}$ to some category $\mathcal{D}$.

Definition 2.23. Let $\mathcal{C}$ be a category. Then a contravariant functor from $\mathcal{C}$ to some category $\mathcal{D}$ is a functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}$.

We also call functors $F : \mathcal{C} \to \mathcal{D}$ *covariant* functors from $\mathcal{C}$ to $\mathcal{D}$.

Definition 2.24. Suppose now that we have two categories $\mathcal{C} \to \mathcal{D}$. Then we may define the *product category* $\mathcal{C} \times \mathcal{D}$ as follows:

- Objects are pairs (C, D) where C is a $\mathcal{C}$-object and D is a $\mathcal{D}$-object.
- Arrows are pairs (f, g) where $f : C \to C'$ is a $\mathcal{C}$-arrow and $g : D \to D'$ is a $\mathcal{D}$-arrow. The domain of (f, g) is (C, D) and the codomain is (C', D').
- Composition is given by componentwise composition. That is, it is given by $(g_{\mathcal{C}}, g_{\mathcal{D}}) \circ (f_{\mathcal{C}}, f_{\mathcal{D}}) = (g_{\mathcal{C}} \circ f_{\mathcal{C}}, g_{\mathcal{D}} \circ f_{\mathcal{D}})$.
- Identities are componentwise identities. That is, for some object (C, D) the corresponding identity $1_{(C,D)} = (1_C, 1_D)$.

It is readily shown that the above indeed gives us a category.

2.6 Construction of new functors from old functors

We have constructions of functors corresponding the construction of new categories above. The first corresponds to taking the opposite category.

Lemma 2.25. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then there is a functor $F^{\text{op}} : \mathcal{C}^{\text{op}} \to \mathcal{D}^{\text{op}}$ defined by $F^{\text{op}}(C) = C$ for objects and $F^{\text{op}}(\overline{f}) = \overline{F(f)}$ for arrows.

Proof. Let $\overline{f} : C' \to C$ be a $\mathcal{C}^{\text{op}}$-arrow. Then $f : C \to C'$ and therefore $F(f) : F(C) \to F(C')$. Consequently $\overline{F(f)} : F(C') \to F(C)$. By definition of F^{op} we now get $F^{\text{op}}(f) : F^{\text{op}}(C') \to F^{\text{op}}(C)$.

Let now $C'' \xrightarrow{\overline{f}} C' \xrightarrow{\overline{g}} C$ be $\mathcal{C}^{\text{op}}$-arrows. Then $C \xrightarrow{g} C' \xrightarrow{f} C''$ in $\mathcal{C}$. Then we have

$$
\begin{aligned}
F^{\text{op}}(\text{Id}_C) &= \overline{F(\text{Id}_C)} = \overline{\text{Id}_{FC}} = \text{Id}_{FC} = \text{Id}_{F^{\text{op}}(C)} \\
F^{\text{op}}(\overline{g} \circ \overline{f}) &= F^{\text{op}}(\overline{f \circ g}) = \overline{F(f \circ g)} = \overline{F(f) \circ F(g)} = \overline{F(g)} \circ \overline{F(f)} = F^{\text{opp}}(\overline{g}) \circ F^{\text{op}}(\overline{f})
\end{aligned}
$$

We conclude that F^{op} is a functor $\mathcal{C}^{\text{op}} \to \mathcal{D}^{\text{op}}$. $\square$

We call the functor from the result above the *opposite functor* of F.

Theorem 2.26. Let $\mathcal{C}$ be a locally small category. Then there is a functor $\text{Hom}_{\mathcal{C}}(-, -) : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \textbf{Set}$ defined $(A, B) \mapsto \text{Hom}_{\mathcal{C}}(A, B)$ on objects and $\text{Hom}_{\mathcal{C}}(-, -)(\overline{f}, g)(h) = g \circ h \circ f$ on arrows (note that f is from A' to A in $\mathcal{C}$ here)

Proof. We verify the functoriality axioms. Let $(A, B) \xrightarrow{(\overline{f}, g)} (A', B') \xrightarrow{(\overline{f'}, g')}$ be composable arrows in $\mathcal{C}^{\text{op}} \times \mathcal{C}$. Then we find for $h \in \text{Hom}_{\mathcal{C}}(A, B)$ that

$$\begin{aligned}
\text{Hom}_{\mathcal{C}}(-,-)(\text{Id}_{(A,B)})(h) &= \text{Hom}_{\mathcal{C}}(-,-)(\text{Id}_A, \text{Id}_B)(h) \\
&= \text{Id}_B \circ h \circ \text{Id}_A \\
&= h \\
&= \text{Id}_{\text{Hom}_{\mathcal{C}}(A,B)}(h) \\
\text{Hom}_{\mathcal{C}}(-,-)((\overline{f'}, g') \circ (\overline{f}, g))(h) &= \text{Hom}_{\mathcal{C}}(-,-)(\overline{f'} \circ \overline{f}, g' \circ g)(h) \\
&= \text{Hom}_{\mathcal{C}}(-,-)(\overline{f \circ f'}, g' \circ g)(h) \\
&= (g' \circ g) \circ h \circ (f \circ f') \\
&= g' \circ (g \circ h \circ f) \circ f' \\
&= \text{Hom}_{\mathcal{C}}(-,-)(\overline{f'}, g')(g \circ h \circ f) \\
&= \text{Hom}_{\mathcal{C}}(-,-)(\overline{f'}, g')(\text{Hom}_{\mathcal{C}}(-,-)(\overline{f}, g)(h)) \\
&= (\text{Hom}_{\mathcal{C}}(-,-)(\overline{f'}, g') \circ \text{Hom}_{\mathcal{C}}(-,-)(\overline{f}, g))(h)
\end{aligned}$$

so $\text{Hom}_{\mathcal{C}}(-,-)$ respects identities and compositions. □

Notation 2.27. We also write $\mathcal{C}(-,-)$ for this functor, or $\text{Hom}(-,-)$ when the category can be inferred.

2.7 Functor categories

Definition 2.28. Let $\mathcal{C}, \mathcal{D}$ be categories and with $\mathcal{C}$ small. We define the *functor category* $[\mathcal{C}, \mathcal{D}]$ with functors from $\mathcal{C}$ to $\mathcal{D}$ as objects, and $\text{Hom}_{[\mathcal{C},\mathcal{D}]}(F, G) = \text{Nat}(F, G)$, the class of natural transformations from F to G. Composition is given by $\mu \circ \eta := ((\mu \circ \eta)_C = \mu_C \circ \eta_C | C \in \text{Obj}(\mathcal{C}))$ and identities by $\text{Id}_F := (\text{Id}_{FC} | C \in \text{Obj}(\mathcal{C}))$.

Lemma 2.29. $[\mathcal{C}, \mathcal{D}]$ is a well-defined category.

Proof. Let $\eta : F \Rightarrow G, \mu : G \Rightarrow H$ be natural transformations between functors $F, G, H : \mathcal{C} \to \mathcal{D}$. It is clear that for $C \in \text{Obj}(\mathcal{C})$ we have $(\mu \circ \eta)_C : F(C) \to H(C)$. Let now $f : C \to C'$ be a $\mathcal{C}$-arrow. Then using the naturality of η, μ we find that the inner squares of

$$\begin{tikzcd}
FC \arrow[r, "\eta_C"] \arrow[d, "F(f)"'] & GC \arrow[r, "\mu_C"] \arrow[d, "G(f)"] & HC \arrow[d, "H(f)"] \\
FC' \arrow[r, "\eta_{C'}"'] & GC' \arrow[r, "\mu_{C'}"'] & HC'
\end{tikzcd}$$

commute and consequently the outer square does too. But the outer square is the naturality square for $\mu \circ \eta$ at f, so $\mu \circ \eta$ is a natural transformation $F \Rightarrow H$, and hence the composition operation is well-defined. As composition of arrows in $\mathcal{D}$ is associative it follows that this composition is (because it is defined pointwise). It remains to verify that Id_F is a natural transformation for any $F : \mathcal{C} \to \mathcal{D}$, and that it satisfies the identity axioms. For an arrow $f : C \to C'$, the naturality square for Id_F is

$$\begin{tikzcd}
FC \arrow[r, "(\mathrm{Id}_F)_C"] \arrow[d, "Ff"'] & FC \arrow[d, "Ff"] \\
FC' \arrow[r, "(\mathrm{Id}_F)_{C'}"] & FC'
\end{tikzcd}$$

which commutes, since the horizontal arrows are identities. It is readily shown that Id_F acts as the identity with respect to composition of natural-transformations. We conclude that $[\mathcal{C}, \mathcal{D}]$ is indeed a category. □

Notation 2.30. We also write $\mathcal{D}^{\mathcal{C}}$ for $[\mathcal{C}, \mathcal{D}]$.

Definition 2.31. A natural transformation $\mu : F \Rightarrow G$ is called a natural isomorphism if it is an isomorphism in the functor category.
If $\mu : F \Rightarrow G$ is a natural isomorphism, we say that F and G are naturally isomorphic.

Proposition 2.32. Let $F, G : \mathcal{C} \to \mathcal{D}$ be funtors and $\mu : F \Rightarrow G$ a natural transformation. Then μ is a natural isomorphism iff each component $\mu_C : F(C) \to G(C)$ is an isomorphism (C in $\mathrm{Obj}(\mathcal{C})$).

Proof. ($\Rightarrow$) Suppose $\mu : F \Rightarrow G$ is a natural isomorphism. Note that the identity natural tranformation is the identity map at each component, and so if $\eta : G \Rightarrow F$ is the inverse of μ, we will have that each component μ_C has η_C as its inverse.
($\Leftarrow$) Conversely suppose each for each $C \in Ob(\mathcal{C})$, $\mu_C : F(C) \to G(C)$ is an isomorphism. Then we define $\eta : G \Rightarrow F$ where each component η_c is defined to be the inverse of μ_C. We still need to verify the naturality condition for η. Let $f : A \to A'$. Consider the following diagram:

$$\begin{tikzcd}
F(A) \arrow[r, "\mu_A"] \arrow[d, "F(f)"'] & G(A) \arrow[r, "\eta_A"] \arrow[d, "G(f)"] & F(A) \arrow[d, "F(f)"] \\
F(A') \arrow[r, "\mu_{A'}"'] & G(A') \arrow[r, "\eta_{A'}"'] & F(A')
\end{tikzcd}$$

Note that the left square commutes by naturality of μ, and the outer rectangle commutes as well, since $\eta_C = (\mu_C)^{-1}$ for each $C \in \mathcal{C}$. So what we get is $F(f) \circ \eta_A \circ \mu_A = \eta_{A'} \circ G(f) \circ \mu_A$, and since μ_A is an isomorphism, we get $F(f) \circ \eta_A = \eta_{A'} \circ G(f)$. Thus, the right square commutes and so η is a natural transformation, and is the inverse of $\mu : F \Rightarrow G$ in the functor category $[\mathcal{C}, \mathcal{D}]$.

□

2.8 Limits and Colimits

In this section we shall discuss limits and colimits. We start by defining initial and terminal objects.

Definition 2.33. Let $\mathcal{C}$ be a category and let X be an object of $\mathcal{C}$. We say that X is an *initial object* in $\mathcal{C}$ if for every object A in $\mathcal{C}$ there is precisely one morphism $f : X \to A$. Furthermore, we say that X is a *terminal object* in $\mathcal{C}$ if for every object A in $\mathcal{C}$ there is precisely one morphism $f : A \to X$.

Example 2.34. An initial object in the category of sets is the empty set $\emptyset$, since for any set A, one may only consider the unique function $f : \emptyset \to A$. An example of a terminal object is the set $\{\emptyset\}$ containing only the empty set, as for any set A there is only the trivial function $f : A \to \{\emptyset\}$ with $f(a) = \emptyset$ for all $a \in A$.

Having defined terminal objects we wish to define cones, such that we can later combine these two notions in the definition of a limit.

Definition 2.35. Let $\mathcal{I}$ and $\mathcal{C}$ be categories and consider the functor $F : \mathcal{I} \to \mathcal{C}$. We then define a *cone* of $\mathcal{C}$ to be an object C of $\mathcal{C}$ together with morphisms $\psi_i : C \to F(i)$ for each indexing object i in $\mathcal{I}$ such that for every morphism $f \in \mathcal{I}(i, j)$ we have $F(f) \circ \psi_i = \psi_j$. We shall write (C, ψ_i) for such a cone.

$$\begin{tikzcd} & C \arrow[dl, "\psi_i"'] \arrow[dr, "\psi_j"] & \\ F(i) \arrow[rr, "F(f)"] & & F(j) \end{tikzcd}$$

Using the above definition we may define the category of cones corresponding to the functor F to be $\mathrm{Cone}(F)$. The objects of this category are the cones corresponding to F and, if (C, ψ_i) and (D, ϕ_i) are two cones, its arrows are the morphisms $f : C \to D$ in $\mathcal{C}$ such that $\phi_i \circ f = \psi_i$ for every $i \in \mathcal{I}$. Having defined the category of cones we may now define a limit of a functor as follows.

Definition 2.36. Let $F : \mathcal{I} \to \mathcal{C}$ be a functor between two categories. A *limit* of F is a terminal object in the cone category $\mathrm{Cone}(F)$. In other words, a *limit* of F is a cone (L, ψ_i) such that for any other cone (C, ϕ_i) there is exactly one morphism $f : (C, \phi_i) \to (L, \psi_i)$ in $\mathrm{Cone}(F)$.

The following definition gives an example of a limit.

Definition 2.37. Let $\mathcal{C}$ be a category and let $F : \{1, 2\} \to \mathcal{C}$ be a functor between the discrete category of two objects and $\mathcal{C}$. Then a *product* is a limit of F.

Similar to a cone, we may define its dual notion, a co-cone, in order to define colimits.

Definition 2.38. Let $F : \mathcal{I} \to \mathcal{C}$ be a functor between two categories. A *co-cone* of F is an object C of $\mathcal{C}$ together with morphisms $\psi_i : F(i) \to C$ such that for every morphism $f : i \to j$ we have $\psi_j \circ F(f) = \psi_i$. We shall write (C, ψ_i) for such a co-cone.

$$\begin{tikzcd} F(i) \arrow[rr, "F(f)"] \arrow[dr, "\psi_i"] & & F(j) \arrow[dl, "\psi_j"] \\ & C & \end{tikzcd}$$

Morphisms between co-cones may then be defined in the same way as was done with cones. We are now ready to define the dual notion of a limit, the colimit.

Definition 2.39. Let $F : \mathcal{I} \to \mathcal{C}$ be a functor between two categories. A *colimit* of F is a co-cone (L, ψ_i) such that for any other co-cone (C, ϕ_i) of F there is a unique morphism $f : (L, \psi_i) \to (C, \phi_i)$ such that $f \circ \psi_i = \phi_i$.

As an example of a colimit we have the dual notion of a product, a co-product.

Definition 2.40. Let $\mathcal{C}$ be a category and let $F : \{1, 2\} \to \mathcal{C}$ be a functor between the discrete category of two elements and $\mathcal{C}$. Then we define a *co-product* to be a colimit of F.

Recall from Lemma 2.25 that a functor $F : \mathcal{C} \to \mathcal{D}$ gives rise to a functor $F^{op} : \mathcal{C}^{op} \to \mathcal{D}^{op}$. We show a limit of F is a colimit for the functor F^{op}.

Proposition 2.41. A limit in a category $\mathcal{C}$ is a colimit in the opposite category $\mathcal{C}^{op}$.

Proof. Let $F : \mathcal{I} \to \mathcal{C}$ be a functor and (Z, μ_i) be a limit for the functor F. We show that $(Z, \overline{\mu_i})$ is a colimit for the functor $F^{op} : \mathcal{I}^{op} \to \mathcal{C}^{op}$. For every $i \in I$, we have $\mu_i : Z \to F(i)$ and so $\overline{\mu_i} : F(i) \to Z$ is a morphism in $\mathcal{C}^{op}$. Also for any $\overline{f} : j \to i$ in $\mathcal{I}^{op}$, we have that the diagram below on the left commutes in $\mathcal{C}$ since (Z, μ_i) is a cone for F. Hence the diagram below on the right commutes in $\mathcal{C}^{op}$.

$$
\begin{tikzcd}
 & Z \arrow[dl, "\mu_i"'] \arrow[dr, "\mu_j"] & \\
F(i) \arrow[rr, "F(f)"'] & & F(j)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & Z & \\
F(i) \arrow[ur, "\overline{\mu_i}"] & & F(j) \arrow[ul, "\overline{\mu_j}"'] \arrow[ll, "\overline{Ff}"]
\end{tikzcd}
$$

Thus, $(Z, \overline{\mu_i})$ is a co-cone for $F^{op} : \mathcal{I}^{op} \to \mathcal{C}^{op}$. To show it is a colimit of F^{op}, suppose $(W, \overline{\psi_i})$ is any other co-cone of F^{op}. Then (W, ψ_i) will be a cone for F and so there exists a unique map $k : W \to Z$ such that for any $i \in \mathcal{I}$, we have the diagram below on the left commutes in $\mathcal{C}$. Hence, the diagram below on the right commutes in $\mathcal{C}^{op}$.

$$
\begin{tikzcd}
 & W \arrow[d, dashed, "k"] \arrow[dl, bend right, "\psi_i"'] \\
 & Z \arrow[dl, "\mu_i"] \\
F(i) &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & W \\
 & Z \arrow[u, dashed, "\overline{k}"'] \\
F(i) \arrow[uur, bend left, "\overline{\psi_i}"] \arrow[ur, "\overline{\mu_i}"'] &
\end{tikzcd}
$$

The uniqueness of $\overline{k}$ follows from the uniqueness of k. Thus, $(Z, \overline{\mu_i})$ is a co-limit of F^{op} in $\mathcal{C}^{op}$. $\square$

Definition 2.42. We say that a category $\mathcal{C}$ has limits of shape $\mathcal{I}$ if a limit exists for each functor $M : \mathcal{I} \to \mathcal{C}$.

Definition 2.43. We say that a category $\mathcal{C}$ has co-limits of shape $\mathcal{I}$ if a co-limit exists for each functor $M : \mathcal{I} \to \mathcal{C}$.

Definition 2.44. Let (C, μ_i) be a limit for a functor $M : \mathcal{I} \to \mathcal{C}$. We say that this limit is preserved by a functor $F : \mathcal{C} \to \mathcal{D}$ if $(F(C), F(\mu_i))$ is a limit for $FM : \mathcal{I} \to \mathcal{D}$ in $\mathcal{D}$. We say that $F : \mathcal{C} \to \mathcal{D}$ preserves limits of shape $\mathcal{I}$ if it preserves any limit for any functor $M : \mathcal{I} \to \mathcal{C}$

Definition 2.45. Let (C, μ_i) be a co-limit for a functor $M : \mathcal{I} \to \mathcal{C}$. We say that this co-limit is preserved by a functor $F : \mathcal{C} \to \mathcal{D}$ if $(F(C), F(\mu_i))$ is a co-limit for $FM : \mathcal{I} \to \mathcal{D}$ in $\mathcal{D}$. We say that $F : \mathcal{C} \to \mathcal{D}$ preserves co-limits of shape $\mathcal{I}$ if it preserves any co-limit for any functor $M : \mathcal{I} \to \mathcal{C}$

3 Representable Functors

As a preview of the upcoming section, we first take a look at the following functor: $\mathcal{O}$: $\mathbf{Top}^{op} \to \mathbf{Set}$ that sends a space X to the set of opens in X, and a continuous map $f : X \to Y$ to $\mathcal{O}(f) : \mathcal{O}(Y) \to \mathcal{O}(X)$, which sends an open set Z in Y to $f^{-1}(Z)$ in $\mathcal{O}(X)$.
Define the Sierpinski space to be the topological space $S = \{0, 1\}$ with the opens being ϕ, S and $\{1\}$.
Let Z be any topological space and $U \subset Z$ an open subset. The function $f_U : Z \to S$ is defined such that

$$f_U(x) = \begin{cases} 0, & x \notin U \\ 1, & x \in U \end{cases}$$

is a continuous map associated to U; conversely, to a continuous $f : Z \to S$ we associate the open subset $U_f := f^{-1}(\{1\})$. This implements a natural bijection $\mathcal{O}(Z) \cong \mathbf{Top}(Z, S)$, where $\mathcal{O}(Z)$ represents the sets of opens in Z and $\mathbf{Top}(Z, S)$ denotes the set of continuous maps from Z to S.
Furthermore the Sierpinski Space is the only topology (up to homeomorphism) on the two-point set S such that the set of opens in a topological space Z is in bijective correspondence with $\mathbf{Top}(Z, S)$.
Suppose we endow $S = \{0, 1\}$ with the discrete topology. Let Z be a topological space such that $U \subset Z$ is open but $Z \setminus U$ is not open. Consider the function f_U as described above. Then f_U is not continuous since $Z \setminus U$ is not open in Z. Thus, $\mathcal{O}(Z)$ is not in bijection with $\mathbf{Top}(Z, S)$. If instead we put the indiscrete topology on $S = \{0, 1\}$, then the set $\mathcal{O}(S)$ has only two elements but every function from $S \to S$ is continuous and so the set $\mathbf{Top}(S, S)$ has 4 elements.
What we have shown is that the Sierpinski topology is the only topology that can be put on a two-point set space such that the set of continuous functions from any topological space Z to S is in bijection with the opens in Z. In fact, we can say something stronger using the Yoneda Lemma – that the Sierpinski space is the *only* space (up to homeomorphism) with this property.
To summarise, we see that the functor O amounts to a functor specifying morphisms into a "unique" representing object (the Sierpinski Space). As we shall see, this is part of a general phenomenon.

3.1 Presheaves and copresheaves

Definition 3.1 (Presheaf)**.** Let $\mathcal{C}$ be a category. A *presheaf* on $\mathcal{C}$ is a functor $X : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$. The *category of presheaves* is the functor category $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ (which has natural transformations between presheaves as morphisms).

Notation 3.2. There are multiple notations for the category of presheaves, namely $\widehat{\mathcal{C}}$, $\mathrm{PSh}(\mathcal{C})$, $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, and $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$. $\widehat{\mathcal{C}}$ is a convenient shorthand, but in some contexts it is clearer to recognise it as a functor category (i.e. when comparing it to $\mathbf{Set}^{\mathcal{C}}$).

This is a rather compact definition of presheaves, making use of the concepts of functors, natural transformations and the opposite category, so it is worth giving an equivalent unpacked presentation. A presheaf $X_\bullet$ on $\mathcal{C}$ consists of a $\mathcal{C}$-indexed family of sets $\{X_C\}_{C\in\mathcal{C}}$ (X_C corresponds to $F(C)$) and a right-action on these sets given by morphisms of $\mathcal{C}$. The action of a morphism $f : D \to C$ in $\mathcal{C}$ on the presheaf is a function

$$\begin{aligned} X_C &\longrightarrow X_D \\ x &\longmapsto x \cdot f \end{aligned}$$

where $x \cdot f$ is called the *restriction of* x *along* f. This right-action must satisfy $x \cdot 1_C = x$ and $(x \cdot f) \cdot g = x \cdot (fg)$ for $E \xrightarrow{g} D \xrightarrow{f} C$. We call $x \cdot f$ the *restriction of* x *along* f.
This right-action and its axioms is a rephrasing of the morphism conditions for functors $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$, where $x \cdot f$ is notation for $F(f)(x)$. Note the contravariance of the functor is why we phrase it as a *right*-action, so that the law $F(fg)(x) = F(g)(F(f)(x))$ is naturally suggested in $(x \cdot f) \cdot g = x \cdot (fg)$.

Notation 3.3. When working with multiple presheaves, we can use a subscript in $x \cdot_X f$ to make the presheaf explicit, however it is better in some contexts to leave this implicit. The function $F(f)$, (which is the function $X_C \to X_D$ sending x to $x \cdot f$) can be written f^*, where the upper-star is a conventional reminder of the contravariance of the functor F (f_* is used when a covariant-functor is applied to f).

We describe morphisms of presheaves in this notation. A morphism of presheaves $\eta : X_\bullet \to Y_\bullet$ is a family of functions $\{\eta_C : X_C \to Y_C\}$ which respects the restriction operation, i.e. for $f : D \to C$ and $x \in X_C$, $\eta_D(x \cdot_X f) = \eta_C(x) \cdot_Y F$. As a commutative diagram:

$$\begin{tikzcd} x \arrow[d, mapsto] & X_C \arrow[r, "\eta_C"] \arrow[d, "f^*"'] & Y_C \arrow[d, "f^*"] & \eta_C(x) \arrow[d, mapsto] \\ x \cdot f & X_D \arrow[r, "\eta_D"] & Y_D & \eta_C(x) \cdot f \end{tikzcd} \tag{1}$$

Definition 3.4 (Co-presheaf)**.** A *co-presheaf* is a functor $F : \mathcal{C} \to \mathbf{Set}$ and the category of *co-presheaves* is the functor category $[\mathcal{C}, \mathbf{Set}]$ (also written $\mathbf{Set}^{\mathcal{C}}$). It is the dual notion of presheaves: a co-presheaf on $\mathcal{C}$ is a presheaf on $\mathcal{C}^{\mathrm{op}}$ and a presheaf on $\mathcal{C}$ is a co-presheaf on $\mathcal{C}^{\mathrm{op}}$.

Example 3.5. Let $\mathcal{C}$ be a locally-small category. An object C of $\mathcal{C}$ induces a presheaf $\mathrm{Hom}(-, C)$ and a co-presheaf $\mathrm{Hom}(C, -)$. The presheaf sends an object D of $\mathcal{C}$ to the set $\mathrm{Hom}(D, C)$ and a morphism $f : D \to D'$ to the operation of pre-composition by f

$$\begin{aligned} \mathrm{Hom}(D', C) &\longrightarrow \mathrm{Hom}(D, C) \\ e &\longmapsto e \circ f \end{aligned}$$

(in the right-action notation, $e \cdot f := e \circ f$). Similarly, $\mathrm{Hom}(C, -)$ sends D to $\mathrm{Hom}(C, D)$ and a morphism $f : D \to D'$ to the operation of post-composition by f

$$\begin{aligned} \mathrm{Hom}(D, C) &\longrightarrow \mathrm{Hom}(D', C) \\ e &\longmapsto f \circ e \end{aligned}$$

These are easily verified as functors using the axioms for composition in $\mathcal{C}$. In fact, these functors can be derived from the functor $\mathrm{Hom} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ (see Theorem 2.26) by precomposing with functors that fix the first/second object.

$$\mathcal{C}^{\mathrm{op}} \xrightarrow{(-,C)} \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{\mathrm{Hom}} \mathbf{Set}$$
$$\mathcal{C} \xrightarrow{(C,-)} \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{\mathrm{Hom}} \mathbf{Set}$$

Definition 3.6. We call $\mathrm{Hom}(-,C)$ the functor *co-represented by* C and $\mathrm{Hom}(C,-)$ the functor *represented by* C. A functor $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ which is naturally isomorphic to $\mathrm{Hom}(-,C)$ for some object C is called *co-representable* and we call C a *co-representative of* F. Likewise, a functor $G : \mathcal{C} \to \mathbf{Set}$ is called *representable* if it is naturally isomorphic to $\mathrm{Hom}(C,-)$ for some object C, and we call C a *representative* of G.

Since we can typically infer the appropriate variance of the functor, we can drop the "co"-distinction and use *representable* and *representative* when speaking about both covariant and contravariant functors.

Example 3.7. Consider the forgetful functor $U : \mathbf{Grp} \to \mathbf{Set}$. Any group homomorphism $f : \mathbb{Z} \to G$ is determined uniquely by the value of $f(1)$. Thus, we have that the set $\mathbf{Grp}(\mathbb{Z}, G)$ is in bijection with the set G. The bijection is natural in G and thus we have that the functor U is naturally isomorphic to the functor $\mathbf{Grp}(\mathbb{Z}, -)$ and is thus a representable functor with representative $\mathbb{Z}$.

4 The Yoneda Lemma

The Yoneda Lemma, stated and proved below, is enabled by the fact that elements of the presheaf $\mathrm{Hom}(-, C)$ are morphisms in $\mathcal{C}$, so we can restrict along them in the presheaf. Concretely, $f \in \mathrm{Hom}(D, C)$ is a morphism $f : D \to C$, and restriction along f in $\mathrm{Hom}(-, C)$ is the map $(-) \circ f : \mathrm{Hom}(C, C) \to \mathrm{Hom}(D, C)$. The category axioms guarantee the existence of at least one element of $\mathrm{Hom}(C, C)$, the identity, and we find that $1_C \circ f = f \in \mathrm{Hom}(D, C)$. This means that the representable presheaf $\mathrm{Hom}(-, C)$ is *highly-degenerate*, in the sense that any element of any set in the presheaf can be written as the restriction of the identity along that element.

$$\begin{tikzcd}
h \in \mathrm{Hom}(A,C) & & k \in \mathrm{Hom}(B,C) \\
& 1_C \in \mathrm{Hom}(C,C) \arrow[ul, mapsto, "(-)\circ h"'] \arrow[ur, mapsto, "(-)\circ k"] \arrow[dl, mapsto, "(-)\circ g"] \arrow[dr, mapsto, "(-)\circ f"'] & \\
g \in \mathrm{Hom}(C,C) & & f \in \mathrm{Hom}(D,C)
\end{tikzcd}$$

The Yoneda Lemma amounts to the fact that natural transformations between presheaves must be compatible with their restriction operations, i.e. restricting before or after transforming between presheaves yields the same result (see (1)). Thus for natural transformations out of $\mathrm{Hom}(-, C)$, the image of 1_C determines the image of all other elements.

Theorem 4.1 (Yoneda Lemma)**.** Let $\mathcal{C}$ be a small category, $C \in \mathcal{C}$ and $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ a presheaf. There is a bijection

$$\begin{aligned}
\Phi_{C,F} \ : \ \mathrm{Nat}(\mathrm{Hom}(-, C), F) &\xrightarrow{\cong} F(C) \\
\alpha &\longmapsto \alpha_C(1_c).
\end{aligned}$$

Proof. First note that the map is well-defined, since such a natural transformation α at C is a function between sets $\alpha_C : \mathrm{Hom}(C, C) \to F(C)$, which we can apply to the identity on C. For injectivity, we prove that the maps $\alpha_D : \mathrm{Hom}(D, C) \to F(D)$ can be recovered from knowing only $\alpha_C(1_C)$. Let $D \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}(D, C)$. The previous remark identifies that $f = 1_C \circ f = \mathrm{Hom}(-, C)(f)(1_C)$, and naturality of α forces the following diagram to commute.

$$\begin{tikzcd}
1_C \arrow[d, mapsto] & \mathrm{Hom}(C,C) \arrow[r, "\alpha_C"] \arrow[d, "(-)\circ f"'] & F(C) \arrow[d, "F(f)"] \\
f & \mathrm{Hom}(D,C) \arrow[r, "\alpha_D"] & F(D)
\end{tikzcd}$$

Thus we can compute $\alpha_D(f) = F(f)(\alpha_C(1_C))$, and therefore any other natural transformation which agrees with α on 1_C must agree with α everywhere.
For surjectivity, we can invert this process, using the diagram to extend a given choice $1_C \mapsto a \in F(C)$ to a natural map, i.e. $\alpha_D(f : D \to C) := F(f)(a)$. Naturality must be verified

with respect to *any* morphism $g : D' \to D$ (not just morphisms into C). This amounts to (contra) functoriality of F, since $\alpha_{D'}(f \circ g) = F(fg)(a) = Fg(Ff(a)) = Fg(\alpha_D(f))$. The relevant naturality square is the middle square below (the upper and lower squares define α_D and $\alpha_{D'}$.

$$\begin{tikzcd}
1_C \arrow[d, mapsto] & \mathrm{Hom}(C,C) \arrow[r, "\alpha_C"] \arrow[d, "(-)\circ f"'] & F(C) \arrow[d, "F(f)"] & a \arrow[d, mapsto] \\
f \arrow[d, mapsto] & \mathrm{Hom}(D,C) \arrow[r, "\alpha_D"] \arrow[d, "(-)\circ g"'] & F(D) \arrow[d, "F(g)"] & F(f)(a) \\
fg & \mathrm{Hom}(D',C) \arrow[r, "\alpha_{D'}"] & F(D') & F(fg)(a) \\
1_C \arrow[u, mapsto] & \mathrm{Hom}(C,C)) \arrow[r, "\alpha_C"] \arrow[u, "(-)\circ fg"] & F(C) \arrow[u, "F(fg)"'] & a \arrow[u, mapsto]
\end{tikzcd}$$

$\square$

Remark 4.2. This bijection is not simply a coincidence of cardinality, but a prescribed assignment $\mathrm{Nat}(\mathrm{Hom}(-,C),F) \to F(C)$ parametrised by a choice of a presheaf F and an object $C \in \mathcal{C}$. The full statement of the Yoneda Lemma identifies that this parametrised family of maps is natural in the choice of C and F, meaning that morphisms $f : C \to C'$ and natural transformations $\eta : F \Rightarrow G$ induce commutative diagrams relating $\Phi_{C,F}$ to $\Phi_{C',F}$ and $\Phi_{C,G}$. To state this properly requires defining appropriate functors $\mathcal{C}^{\mathrm{op}} \times \widehat{\mathcal{C}} \to \mathbf{Set}$ so that $\Phi_{C,F}$ becomes a natural isomorphism. We delay this until section 6, and in section 8 we demonstrate how this enables us to work with presheaves categorically.

Remark 4.3. The Yoneda Lemma given above is about functors $\mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ (i.e. contravariant functors from $\mathcal{C}$ to $\mathbf{Set}$), but we can use duality to conclude the corresponding statement for covariant functors $\mathcal{C} \to \mathbf{Set}$. Replacing $\mathcal{C}$ by $\mathcal{C}^{\mathrm{op}}$, the presheaf $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(-,C) : (\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \to \mathbf{Set}$ is really just $\mathrm{Hom}_{\mathcal{C}}(C,-) : \mathcal{C} \to \mathbf{Set}$, since $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(D,C) = \mathrm{Hom}_{\mathcal{C}}(C,D)$ and pre-composition in $\mathcal{C}^{\mathrm{op}}$ is just post-composition in $\mathcal{C}$. Similarly, a presheaf $F : (\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \to \mathbf{Set}$ is just a co-presheaf $F : \mathcal{C} \to \mathbf{Set}$.
We obtain what we refer to as the *covariant Yoneda Lemma*, and we can also refer to Theorem 4.1 as the *contravariant Yoneda Lemma*.

Corollary 4.4 ((Covariant) Yoneda Lemma)**.** Let $\mathcal{C}$ be a small category, $C \in \mathcal{C}$ and $F : \mathcal{C} \to \mathbf{Set}$ a co-presheaf. There is a bijection

$$\begin{aligned} \Phi_{C,F} \ : \ \mathrm{Nat}(\mathrm{Hom}(C,-),F) &\xrightarrow{\ \cong\ } F(C) \\ \alpha &\longmapsto \alpha_C(1_c). \end{aligned}$$

5 Examples of the Yoneda correspondence

We explore the correspondence $\mathrm{Nat}(\mathrm{Hom}(-,C),F) \cong F(C)$ via examples. By specialising the choice of a category $\mathcal{C}$, we aim to recognise $\mathrm{Nat}(\mathrm{Hom}(-,C),F)$ as some familiar concept, and then re-prove the correspondence to $F(C)$ in this setting.
A natural way to seek out simplifications is to restrict to less-general categories. The concept of a category can be seen as the common generalisation of monoids and pre-orders. Specifically there are embeddings **Mon** $\hookrightarrow$ **Cat** and **Pre** $\hookrightarrow$ **Cat** realising a monoid as a category with a single object, and a pre-order as an *ordered category* i.e., one with at most one morphism between any two objects. Thus we begin by investigating presheaves on monoids and pre-orders (considered as categories).

5.1 Presheaves on a monoid

Definition 5.1. Let M be a monoid with a binary operation written multiplicatively (i.e. $(m,n) \mapsto mn$). The category BM consists of a single object $*$, and the hom-set $BM(*,*) := M$ is defined to be the underlying set of the monoid. The identity Id_* is defined to be the identity e of the monoid, and composition is given by the associative monoid operation. It is clear that this satisfies the definition of a category, since the category axioms reduce to the same statements as the axioms for a monoid (with elements of the monoid corresponding to the morphisms), when only one object is involved.

Example 5.2. Let M be a monoid regarded as an one-object category BM. A functor $F : BM^{\mathrm{op}} \to \mathbf{Set}$ consists of a set S which is the value of the functor at the unique object of BM^{op}, along with, for each morphism $m \in BM^{\mathrm{op}}$, a map $F(m) : S \to S$ satisfying the functoriality axioms. For each $m \in M$ and $s \in S$, we write $F(m)(s) = s \cdot m$, and so the functor F amounts to a set S and a function $f : S \times M \to S$ that sends (s,m) to $s \cdot m$ satisfying:

- $s \cdot e = s$ and
- $s \cdot (m_1 m_2) = (s \cdot m_1) \cdot m_2$ for every $s \in S$ and $m_1, m_2 \in M$.

Thus, a functor $F : BM^{\mathrm{op}} \to \mathbf{Set}$ corresponds to a right M-set.

The category BM^{op} has a unique object $*$ and so we only have one representable functor from BM^{op}, and we write it as $\underline{M} = BM(\text{-},*) : BM^{\mathrm{op}} \to \mathbf{Set}$ which sends the unique object $*$ to $BM(*,*) = M$ and a morphism $m : * \to * \in M$ to the function $f_m : M \to M$ that sends an element $x \in M$ to $xm \in M$.

Let X be a right M-set. Under the identification of functors from $BM^{\mathrm{op}} \to \mathbf{Set}$ as described in the first paragaraph above, a natural transformation $\phi : \underline{M} \to X$ comprises of a unique component $\phi : M \to X$, and the naturality condition gives that for any $m : * \to *$, we have that the following square commutes:

$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[d, "(-)\cdot m"'] & X \arrow[d, "(-)\cdot m"] \\ M \arrow[r, "\phi"'] & X \end{tikzcd}$$

where the vertical maps denotes the right-action of m in the respective M-sets.
Or in other words, for every $x \in M$, $\phi(xm) = \phi(x) \cdot m$ (such a map is called a map of M-sets). Further, note that any such map $\alpha : \underline{M} \to X$ is uniquely determined by its value on the identity $e \in M$. Since for any $x \in M$, $\alpha(x) = \alpha(e \cdot x) = \alpha(e) \cdot x$. This gives us a bijection between $\mathrm{Nat}(\underline{M}, X)$ and the elements of X.

Next we prove that the bijection $\phi_X : \mathrm{Nat}(\underline{M}, X) \to X(*)$ which sends $\alpha : M \to X$ to $\alpha(e) \in X$ is natural in X i.e. if $X, X' : BM^{op} \to \mathbf{Set}$ and $\mu : X \Rightarrow X'$ a natural transformation between them, then the following diagram commutes:

$$\begin{tikzcd} \mathrm{Nat}(\underline{M}, X) \arrow[r, "\phi_X"] \arrow[d, "\mu\circ-"'] & X(*) \arrow[d, "\mu(*)"] \\ \mathrm{Nat}(\underline{M}, X') \arrow[r, "\phi_{X'}"'] & X'(*) \end{tikzcd}$$

Let $\alpha : \underline{M} \to X$ be a map of M-sets. Then $\mu(*) \circ \phi_X(\alpha) = \mu(*) \circ (\alpha(e)) = \mu(\alpha(e))$ and, $\phi_{X'}(\mu \circ \alpha) = \mu(\alpha(e))$. This proves that the square above commutes.
Hence, in the case of diagrams indexed by a monoid BM, natural transformations whose domain is a representable functor are determined by the choice of a single element. And the element is obtained by evaluating the codomain functor at the representing object.

Remark: In fact a similar argument can be used to prove that the Yoneda Lemma holds true for any one-object category. For example we can regard groups as small one-object categories where each arrow has an inverse.

5.2 Presheaves on pre-ordered categories

For a set X, a *preordering* on X is a relation $\preceq$ on X which is *reflexive* and *transitive*, i.e.

(i) $\forall x \in X, x \preceq x$ (reflexivity)

(ii) $\forall x, y, z \in X, x \preceq y \wedge y \preceq z \Rightarrow x \preceq z$ (transitivity).

The pair $(X, \preceq)$ is called a *preorder* or a *preordered set*. The class of preorders can be promoted to a category by considering *order-preserving functions*, i.e. a morphism $f : (X, \preceq) \to (Y, \leq)$ of preorders is a function $f : X \to Y$ such that $x \preceq y \Rightarrow f(x) \leq f(y)$ (identities and composition are the same as in **Set**). It is trivial to check that identities are order-preserving, and that the composition of order-preserving functions is order-preserving. We call this category **Pre**.

A category can be viewed as a generalisation of the notion of a preorder. We can view $(X, \preceq)$ as a category, with $\mathrm{Obj}(X, \preceq) := X$, and for $x, y \in X$

$$\mathrm{Hom}_{(X,\preceq)}(x, y) := \begin{cases} \{(x, y)\} & , \ x \preceq y \\ \emptyset & , \text{otherwise.} \end{cases}$$

The composition operation is given uniquely by $(y, z) \circ (x, y) = (x, z)$, and identities by $1_x := (x, x)$. The reflexivity and transitivity axioms ensure that these are well-defined, and that the associativity and identity axioms are respected. In fact this construction is part of an embedding $\mathbf{Pre} \hookrightarrow \mathbf{Cat}$, and this is the formal sense in which categories generalise pre-orders.
We give an example of a presheaf on such a pre-order category, in the more typical scenario where the ordering satisfies the stronger condition of being a *partial-ordering*, i.e. it is anti-symmetric: $\forall x, y \in X, x \preceq y \wedge y \preceq x \Rightarrow x = y$.

5.3 Presheaves on a topological space

Let X be a topological space, with topology $\mathcal{T}(X) \subseteq \mathscr{P}(X)$. The topology $\mathcal{T}(X)$ inherits the partial ordering $\subseteq$ from $\mathscr{P}(X)$, and so $\mathcal{T}(X)$ can be regarded as a category by the previous construction.

Definition 5.3. A presheaf on the topological space X is a presheaf $\mathcal{F} : \mathcal{T}(X)^{\mathrm{op}} \to \mathbf{Set}$ on the category $\mathcal{T}(X)$. Since a pair of opens (U, V) determines at most one morphism $U \to V$ in $\mathcal{T}(X)$, we write the action of $\mathcal{F}$ on such a morphism as $\mathcal{F}(U \subseteq V)$.

Example 5.4. The prototypical example of such a presheaf is the functor $\mathrm{Cts}(-, \mathbb{R})$ (or fix any other space in place of $\mathbb{R}$). At the open set U, this is the set of continuous maps $f : U \to \mathbb{R}$. The unique morphism for $U \subseteq V$ corresponds to the restriction operation $\mathrm{Cts}(V, \mathbb{R}) \to \mathrm{Cts}(U, \mathbb{R})$ sending $f : V \to \mathbb{R}$ to its restriction $f|_U : U \to \mathbb{R}$.

Example 5.5. For an open set W, the representable presheaf $h_W : \mathcal{T}(X)^{\mathrm{op}} \to \mathbf{Set}$ sends $U \subset W$ to a singleton set, and all other opens U (not a subset of W) to the empty set. Thus a natural transformation $\alpha : h_W \Rightarrow \mathrm{Cts}(-, \mathbb{R})$ is a family $\{f_U\}_U$ of choices $f_U := \alpha_U(*) \in \mathrm{Cts}(U, \mathbb{R})$, satisfying $(f_V)|_U = f_V$ whenever $U \subseteq V$. Clearly this naturality condition can be used to generate such a family from a single continuous map $f : W \to \mathbb{R}$ by defining $f_U := f|_U$, and we can recover the original map as $f_W = f|_W = f$. Thus we have a bijection

$$\begin{aligned} \mathrm{Nat}(h_W, \mathrm{Cts}(-, \mathbb{R})) &\longrightarrow \mathrm{Cts}(W, \mathbb{R}) \\ \{f_U\}_U &\longmapsto f_W. \end{aligned}$$

In fact, this is precisely the bijective correspondence $\alpha \mapsto \alpha_W(1_W) \in \mathrm{Cts}(W, \mathbb{R})$ given by the Yoneda Lemma.

5.4 The category of directed multigraphs as a presheaf category

Let $\mathcal{C}$ be a category that has two objects which we denote by 0 and 1, and two non-identity morphisms $s, t : 0 \to 1$. We claim that the category of presheaves over $\mathcal{C}$ is the category of directed multigraphs with loops.

Let $G : \mathcal{C}^{op} \to \mathbf{Set}$ be a functor. Then G takes objects of $\mathcal{C}$ to sets i.e. $G(0) = V$ and $G(1) = E$ for arbitrary sets V and E. These sets will represent the vertices and edges of our graph G, respectively.

G takes an arrow in $\mathcal{C}$ to a set function but since the functor is contravariant, we get two functions: $G(s) : G(1) = E \to G(0) = V$ and $G(t) : G(1) = E \to G(0) = E$. We interpret $G(s)$ as a function that takes an edge and gives its source vertex and $G(t)$ as a function that takes an edge and gives its target vertex.

Let G, H be two presheaves over $\mathcal{C}$. A natural transformation $\eta : G \Rightarrow H$ comprises of two maps $\eta_0 : G(0) \to H(0)$ and $\eta_1 : G(1) \to H(1)$ such that the following squares commute:

$$\begin{tikzcd} G(1) \arrow[r, "\eta_1"] \arrow[d, "G(s)"'] & H(1) \arrow[d, "H(s)"] \\ G(0) \arrow[r, "\eta_0"'] & H(0) \end{tikzcd} \qquad \begin{tikzcd} G(1) \arrow[r, "\eta_1"] \arrow[d, "G(t)"'] & H(1) \arrow[d, "H(t)"] \\ G(0) \arrow[r, "\eta_0"'] & H(0) \end{tikzcd}$$

Viewing $G(0)$ and $H(0)$ as the vertex set of graphs G and H respectively with edge set $G(1)$ and $H(1)$ respectively, we get that the commutativity of the two squares above tell us that under η_0 the source and target of an edge e must be mapped to the source and target of $\eta_1(e)$ respectively. Thus, a natural transformation $\eta : G \Rightarrow H$ corresponds to a graph morphism $\eta : G \to H$ whose action on the vertex set is given by η_0 and its action on the edge set is given by η_1.

Since there are only two object in $\mathcal{C}$, we have that there are only two representable functors from $\mathcal{C}^{op} \to \mathbf{Set}$, namely, $\mathcal{C}(-, 0)$ and $\mathcal{C}(-, 1)$. The first representable functor $\mathcal{C}(-, 0)$, by our earlier correspondence, represents a graph G with only one vertex and no edges. While the functor $\mathcal{C}(-, 1)$ represents the following directed graph H:

$$v_0 \longrightarrow v_1$$

So, for any presheaf X over $\mathcal{C}$, any natural transformation from $\mathcal{C}(-, 0)$ to X represents a graph morphism between the graph G and the graph corresponding to X. Since G has only one vertex and no edges, the set of graph morphisms from G to X will correspond bijectively to the vertex set $X(0)$ of X. Thus, we have $\mathrm{Nat}(\mathcal{C}(-, 0), X) \cong X(0)$. Similarly, the set of graph morphisms from H to the graph X is in bijective correspondence to the set of edges in X. So we get that $\mathrm{Nat}(\mathcal{C}(-, 1), X) \cong X(1)$. Thus, we have recognised why the Yoneda Lemma holds for this presheaf category.

6 The Yoneda Isomorphism

The Yoneda Lemma as stated in section 4 provides a parametrised family of bijections between the sets $\mathrm{Nat}(\mathrm{Hom}(-,C),F)$ and $F(C)$. However, this can be strengthened to an isomorphism between two **Set**-valued functors, one producing the set $\mathrm{Nat}(\mathrm{Hom}(-,C),F)$ and the other producing $F(C)$ when evaluated at C and F. In this section, we build the necessary structure required to define these functors.

6.1 Constructions

The first construction corresponds to the product of categories.

Lemma 6.1. Let $F : \mathcal{A} \to \mathcal{C}$, $G : \mathcal{B} \to \mathcal{D}$ be functors. Then there is a functor $F \times G : \mathcal{A} \times \mathcal{B} \to \mathcal{C} \times \mathcal{D}$ defined by $(F \times G)(A, B) = (F(A), G(B))$ for objects and $(F \times G)(f, g) = (F(f), G(g))$ for arrows.

Proof. First note that $(F \times G)(f, g) = (F(f), G(g)) : (F(A), G(B)) \to (F(A'), G(B'))$ has the appropriate domain and codomain.

Now suppose we have a composition $(A, B) \xrightarrow{(f_{\mathcal{A}}, f_{\mathcal{B}})} (A', B') \xrightarrow{(g_{\mathcal{A}}, g_{\mathcal{B}})} (A'', B'')$. Then we have

$$\begin{aligned}
(F \times G)((g_{\mathcal{A}}, g_{\mathcal{B}}) \circ (f_{\mathcal{A}}, f_{\mathcal{B}})) &= (F(g_{\mathcal{A}} \circ f_{\mathcal{A}}), G(g_{\mathcal{B}} \circ f_{\mathcal{B}})) \\
&= (F(g_{\mathcal{A}}) \circ F(f_{\mathcal{A}}), G(g_{\mathcal{B}}) \circ G(f_{\mathcal{B}})) \\
&= (F(g_{\mathcal{A}}), G(g_{\mathcal{B}})) \circ (F(f_{\mathcal{A}}), G(f_{\mathcal{B}})) \\
&= (F \times G)(g_{\mathcal{A}}, g_{\mathcal{B}}) \circ (F \times G)(f_{\mathcal{A}}, f_{\mathcal{B}}), \\
(F \times G)(\mathrm{Id}_A, \mathrm{Id}_B) &= (F(\mathrm{Id}_A).G(\mathrm{Id}_B)) \\
&= (\mathrm{Id}_{FA}, \mathrm{Id}_{GA}) \\
&= \mathrm{Id}_{(FA,GA)} \\
&= \mathrm{Id}_{(F \times G)(A,B)} .
\end{aligned}$$

which shows that $F \times G$ is indeed a well-defined functor. This completes the proof. $\square$

The following lemma is useful in constructing functors.

Lemma 6.2. Suppose that $\mathcal{C}$ is some category. Then there is a functor $S : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ defined by $S(A) = (A, A)$ for objects and $S(f) = (f, f)$ for arrows.

Proof. Let $f : A \to B$ be a $\mathcal{C}$-arrow. Then $(f, f) : (A, A) \to (B, B)$, or equivalently $S(f) : S(A) \to S(B)$. Let now $A \xrightarrow{f} B \xrightarrow{g} C$ be $\mathcal{C}$-arrows. Then we find

$$\begin{aligned}
S(\mathrm{Id}_A) &= (\mathrm{Id}_A, \mathrm{Id}_A) \\
&= \mathrm{Id}_{(A,A)} \\
&= \mathrm{Id}_{S(A)}, \\
S(g \circ f) &= (g \circ f, g \circ f) \\
&= (g, g) \circ (f, f) \\
&= S(g) \circ S(f).
\end{aligned}$$

We see that S is indeed a well-defined functor. $\square$

We now get the following corollary.

Corollary 6.3. Let $F : \mathcal{C} \to \mathcal{D}, G : \mathcal{C} \to \mathcal{E}$ be functors. Then there is a functor denoted (F, G) from $\mathcal{C}$ to $\mathcal{D} \times \mathcal{E}$ defined by $(F, G)(A) = (F(A), G(A))$ on objects and $(F, G)(f) = (F(f), G(f))$ on arrows.

Proof. The functor (F, G) is precisely the composition $(F \times G) \circ S$. $\square$

Lemma 6.4. Let $\mathcal{C}, \mathcal{D}$ be locally small categories. Then there is a functor denoted ev : $\mathcal{D} \times \mathcal{C}^{\mathcal{D}} \to \mathcal{C}$ given on objects A by $\mathrm{ev}(C, F) = F(C)$ and on arrows $(f, \eta) : (A, F) \to (B, G)$ by $\mathrm{ev}(f, \eta) = G(f) \circ \eta_A = \eta_B \circ F(f)$.

Proof. It is clear that ev respects domains and codomains. Let now $(A, F) \xrightarrow{(f,\alpha)} (B, G) \xrightarrow{(g,\eta)} (C, H)$ be arrows in $\mathcal{D} \times \mathcal{C}^{\mathcal{D}}$. Then $A \xrightarrow{f} B \xrightarrow{g} C$ are arrows in $\mathcal{D}$ and $F \overset{\alpha}{\Rightarrow} G \overset{\eta}{\Rightarrow} H$ are natural transformation between functors $\mathcal{D} \to \mathcal{C}$. We now find

$$\begin{aligned}
\mathrm{ev}((g, \eta) \circ (f, \alpha)) &= \mathrm{ev}(g \circ f, \eta \circ \alpha) \\
&= H(g \circ f) \circ (\eta \circ \alpha)_A \\
&= H(g) \circ H(f) \circ \eta_A \circ \alpha_A \\
&= H(g) \circ \eta_B \circ G(f) \circ \alpha_A \\
&= \mathrm{ev}(g, \eta) \circ \mathrm{ev}(f, \alpha), \\
\mathrm{ev}(\mathrm{Id}_{(A,F)}) &= \mathrm{ev}(\mathrm{Id}_A, \mathrm{Id}_F) \\
&= F(\mathrm{Id}_A) \circ (\mathrm{Id}_F)_A \\
&= \mathrm{Id}_{FA} \circ \mathrm{Id}_{FA} \\
&= \mathrm{Id}_{FA} \\
&= \mathrm{Id}_{\mathrm{ev}(A,F)} .
\end{aligned}$$

We see that ev is indeed a functor. $\square$

Recall from section 3 that any object C of a locally-small category $\mathcal{C}$ induces two functors (co)-represented by C, namely $\mathrm{Hom}(-, C) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ and $\mathrm{Hom}(C, -) : \mathcal{C} \to \mathbf{Set}$. In fact the construction $C \mapsto \mathrm{Hom}(-, C)$ (resp. $C \mapsto \mathrm{Hom}(C, -)$) can be expanded to a functor $\mathcal{C} \to \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ (resp. $\mathcal{C}^{\mathrm{op}} \to \mathbf{Set}^{\mathcal{C}}$).

Notation 6.5. We introduce the notation $h_C := \mathrm{Hom}(-, C)$ and $h^C := \mathrm{Hom}(C, -)$, so that we can disambiguate the functors $h_{(-)}$ and $h^{(-)}$.

Lemma 6.6. Let $\mathcal{C}$ be a locally small category. Then there is a functor denoted by $h^{(-)}$: $\mathcal{C}^{\mathrm{op}} \to \mathbf{Set}^{\mathcal{C}}$ which sends an object C to h^C and an arrow $\overline{f} : A \to B$ to the natural transformation $h^{\overline{f}} : h^A \Rightarrow h^B$ defined by $(h^{\overline{f}})_C(g) = g \circ f$.

Proof. We first show that the arrow assignment is well-defined. Let $\overline{f} : A \to B$ be a $\mathcal{C}^{\mathrm{op}}$-arrow, C a $\mathcal{C}$-object and let $g \in h^A(C) = \mathrm{Hom}_{\mathcal{C}}(A, C)$. Then $f : B \to A$ and consequently $h \circ f : B \to C$ so $h \circ f \in \mathrm{Hom}_{\mathcal{C}}(B, C) = h^B(C)$. We see that

$(h^{\overline{f}})_C : \mathrm{Hom}_{\mathcal{C}}(A, C) \to \mathrm{Hom}_{\mathcal{C}}(B, C)$ is well-defined for all C. It remains to prove naturality. Let $\overline{f} : A \to B$ be a $\mathcal{C}^{\mathrm{op}}$-arrow, let $g : C \to D$ be a $\mathcal{C}$-arrow and finally let $k \in \mathrm{Hom}(A, C)$. Then we find

$$(h^B(g) \circ (h^{\overline{f}})_C)(k) = h^B(g)(k \circ f) = g \circ k \circ f = (h^{\overline{f}})_D(g \circ k) = ((h^{\overline{f}})_D \circ h^A(g))(k)$$

which proves naturality. This shows that the arrow assignment is well-defined and therefore so is $h^{(-)}$ as a double assignment. It is clear that $h^{(-)}$ respects domains and codomains.

Let now $A \xrightarrow{\overline{f}} B \xrightarrow{\overline{g}} C$ be $\mathcal{C}^{\mathrm{op}}$-arrows. Let D be a $\mathcal{C}$-object. Let finally $k \in \mathrm{Hom}(A, D)$. Then we find

$$(h^{\overline{g}\circ\overline{f}})_D(k) = (h^{\overline{f\circ g}})_D(k) = k \circ (f \circ g) = (h^{\overline{g}})_D(k \circ f) = ((h^{\overline{g}})_D \circ (h^{\overline{f}})_D)(k) = (h^{\overline{g}} \circ h^{\overline{f}})_D(k)$$

so $h^{\overline{g}\circ\overline{f}} = h^{\overline{g}} \circ h^{\overline{f}}$. We see that $h^{(-)}$ is a indeed a functor. $\square$

In an analogous way we can also prove the following

Lemma 6.7. Let $\mathcal{C}$ be a locally small category. Then there is a functor denoted by $h_{(-)}$ from $\mathcal{C}$ to $\hat{\mathcal{C}} = \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ which sends an object C to h_C and an arrow $f : A \to B$ to the natural transformation $h_f = ((h_f)_C | C \text{ a } \mathcal{C}\text{-object}) : h_A \Rightarrow h_B$ defined by $(h_f)_C(g) = f \circ g$.

The functor $h_{(-)} : \mathcal{C} \to \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ is also known as the *Yoneda embedding.*

6.2 The covariant Yoneda Isomorphism

Let $\mathcal{C}$ again be a locally small category. In the preceding chapter we constructed a functor ev from $\mathcal{C} \times \mathbf{Set}^{\mathcal{C}}$ to $\mathbf{Set}$. We may now define a second functor $I : \mathcal{C} \times \mathbf{Set}^{\mathcal{C}} \to \mathbf{Set}$ by the composition

$$\mathcal{C} \times \mathbf{Set}^{\mathcal{C}} \xrightarrow{(h^{(-)})^{\mathrm{op}} \times \mathrm{Id}_{\mathcal{C}}} (\mathbf{Set}^{\mathcal{C}})^{\mathrm{op}} \times \mathbf{Set}^{\mathcal{C}} \xrightarrow{\mathrm{Hom}_{\mathbf{Set}^{\mathcal{C}}}(-,-)} \mathbf{Set}.$$

This functor I acts on objects (C, F) as

$$I(C, F) = (\mathrm{Hom}_{\mathbf{Set}^{\mathcal{C}}}(-,-) \circ ((h^{(-)})^{\mathrm{op}} \times \mathrm{Id}_{\mathcal{C}}))(C, F) = \mathrm{Hom}_{\mathbf{Set}^{\mathcal{C}}}(-,-)(h^C, F) = \mathrm{Nat}(h^C, F)$$

and on arrows $(f, \eta) : (A, F) \to (B, G)$ as

$$I(f, \eta)(\alpha) = \mathrm{Hom}_{\mathbf{Set}^{\mathcal{C}}}(-,-)(h^{\overline{f}}, \eta)(\alpha) = \eta \circ \alpha \circ h^{\overline{f}}$$

for $\alpha : h^A \Rightarrow F$. The Yoneda Lemma precisely says that there is a natural isomorphism Φ between these two functors.

Theorem 6.8 (Covariant Yoneda Isomorphism)**.** Let $\mathcal{C}$ be a locally small category and let ev, I be functors as above. Then there is a natural isomorphism $\Phi : I \to \mathrm{ev}$ given by $\Phi_{C,F}(\alpha) = \alpha_C(\mathrm{Id}_C)$.

Proof. The bijections $\Phi_{C,F}$ are precisely those from section 4. It remains to show that the collection $\Phi = (\Phi_{C,F} : \mathrm{Nat}(h^C, F) \to F(C) \mid (C, F) \in \mathcal{C} \times \mathbf{Set}^{\mathcal{C}})$ is a natural transformation. Let $(f, \eta) : (A, F) \to (B, G)$ be a $\mathcal{C} \times \mathbf{Set}^{\mathcal{C}}$-arrow and $\alpha : h^A \Rightarrow F$ a natural transformation. Then

$$\begin{aligned}
(\Phi_{B,G} \circ I(f, \eta))(\alpha) &= \Phi_{B,G}(\eta \circ \alpha \circ h^{\overline{f}}) \\
&= (\eta \circ \alpha \circ h^{\overline{f}})_B(\mathrm{Id}_B) \\
&= (\eta_B \circ \alpha_B \circ (h^{\overline{f}})_A)(\mathrm{Id}_B) \\
&= \eta_B(\alpha_B(\mathrm{Id}_B \circ f)) \\
&= \eta_B(\alpha_B(f)) \\
&= \eta_B(F(f)(\alpha_A(\mathrm{Id}_A))) \\
&= G(f)(\eta_A(\alpha_A(\mathrm{Id}_A))) \\
&= (G(f) \circ \eta_A)(\Phi_{A,F}(\alpha)) \\
&= (\mathrm{ev}(f, \eta) \circ \Phi_{A,F})(\eta)
\end{aligned}$$

which proves naturality of Φ. This completes the proof. □

The result above has the following important corollary:

Corollary 6.9. Let $\mathcal{C}$ be a locally small category, C be some $\mathcal{C}$-object and $F : \mathcal{C} \to \mathbf{Set}$ a functor. Then there is a bijection $\Phi_{C,F} : \mathrm{Nat}(h^C, F) \to F(C)$ given by $\Phi_{C,F}(\alpha) = \alpha_C(\mathrm{Id}_A)$.

6.3 The contravariant Yoneda Isomorphism

In the preceding chapter we saw the covariant version of the Yoneda Lemma. We will now study the contravariant version. We again have the evaluation functor $\mathrm{ev} : \mathcal{C}^{\mathrm{op}} \times \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}} \to \mathbf{Set}$ and a second functor $J : \mathcal{C}^{\mathrm{op}} \times \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}} \to \mathbf{Set}$ defined by the compostion

$$\mathcal{C}^{\mathrm{op}} \times \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}} \xrightarrow{(h_{(-)})^{\mathrm{op}} \times \mathrm{Id}_{\mathcal{C}^{\mathrm{op}}}} (\mathbf{Set}^{\mathcal{C}}) \times (\mathbf{Set}^{\mathcal{C}})^{\mathrm{op}} \xrightarrow{\mathrm{Hom}_{\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}}(-,-)} \mathbf{Set}$$

This functor J acts on objects (C, F) as

$$J(C, F) = \mathrm{Nat}(h_C, F)$$

and on arrows $(\overline{f}, \eta) : (A, F) \to (B, G)$ as

$$J(\overline{f}, \eta)(\alpha) = \eta \circ \alpha \circ h_f$$

for $\alpha : h_A \Rightarrow F$. We can now formulate the contravariant version of the Yoneda Lemma.

Theorem 6.10 (Contravariant Yoneda Isomorphism)**.** Let $\mathcal{C}$ be a locally small category and let ev, J be functors as above. Then there is a natural isomorphism $\Phi : J \to \mathrm{ev}$ given by $\Phi_{C,F}(\alpha) = \alpha_C(\mathrm{Id}_C)$.

This version is proven similarly to the covariant version. It also has the following corollary:

Corollary 6.11. Let $\mathcal{C}$ be a locally small category, C be some $\mathcal{C}$-object and $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ a a functor (i.e. a contravariant functor from $\mathcal{C}$ to $\mathbf{Set}$). Then there is a bijection $\Phi_{C,F} : \mathrm{Nat}(h_C, F) \to F(C)$ given by $\Phi_{C,F}(\alpha) = \alpha_C(\mathrm{Id}_C)$.

7 Applications

7.1 The Yoneda Embedding

Recall the functor $h_{(-)} : \mathcal{C} \to \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ from section 6.1 which sends C to $\mathrm{Hom}(-, C)$. We prove that this functor is an embedding of the category $\mathcal{C}$ inside the presheaf category $\widehat{\mathcal{C}} = \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$.

Remark 7.1. Although $h_{(-)} : \mathcal{C} \to \widehat{\mathcal{C}}$ and $h^{(-)} : \mathcal{C}^{\mathrm{op}} \to \widehat{\mathcal{C}^{\mathrm{op}}}$ are dual to each-other, we focus on $h_{(-)}$ since it embeds $\mathcal{C}$, rather than the opposite category of $\mathcal{C}$, however the dual results of this section are still useful.

Proposition 7.2 (The Yoneda embedding)**.** The functor $h_{(-)} : \mathcal{C} \to \widehat{\mathcal{C}}$ is an embedding.

Proof. Recall that an embedding is a fully-faithful functor which is injective on objects. $h_{(-)}$ is injective on objects, since if $C \neq D$ then $1_C \in h_C(C)$ but $1_C \notin h_D(C)$, so h_C and h_D are distinct presheaves. That $h_{(-)}$ is fully faithful is an immediate corollary of the Yoneda Lemma (Theorem 4.1).
For objects $C, D \in \mathcal{C}$, we must show that the map

$$\begin{aligned} \mathcal{C}(C, D) &\longrightarrow \widehat{\mathcal{C}}(h_C, h_D) \\ f &\longmapsto h_f := \{g \mapsto f \circ g\} \end{aligned}$$

is a bijection (surjective = full, injective = faithful). Taking $C := C, F := h_D$, the Yoneda Lemma gives a bijection

$$\Phi_{C,h_D} : \mathrm{Nat}(h_C, h_D) \xrightarrow{\cong} h_D(C)$$

But $h_D(C) = \mathcal{C}(C, D)$ and $\mathrm{Nat}(h_C, h_D) = \widehat{\mathcal{C}}(h_C, h_D)$, and this is precisely the inverse of the first map, since $\Phi_{C,h_D}(h_f) = (h_f)_{1_C}(1_C) = f \circ 1_C = f$.

□

Corollary 7.3. Let C, D be objects in a locally-small category $\mathcal{C}$. Then $C \cong D$ in $\mathcal{C}$ if and only if $\mathcal{C}(-, C) \cong \mathcal{C}(-, D)$ in $\widehat{\mathcal{C}}$.

Proof. Fully faithful functors preserve and reflect isomorphisms (see 2.14), so an isomorphism $f : C \to D$ induces an isomorphism $h_f : \mathcal{C}(-, C) \cong \mathcal{C}(-, D)$ and vice versa. □

Corollary 7.4. Let C, D be objects in a locally-small category $\mathcal{C}$. Then $C \cong D$ in $\mathcal{C}$ if and only if $\mathcal{C}(C, -) \cong \mathcal{C}(D, -)$ in $\widehat{\mathcal{C}}$.

Proof. This is the dual statement of the previous corollary (apply the previous corollary to $\mathcal{C}^{\mathrm{op}}$, or redo the proof using the covariant Yoneda Lemma). □

We can state these corollaries without relying on the definitions of (co)-presheaves.

Corollary 7.5. Let $\mathcal{C}$ be a locally-small category and X, Y objects in $\mathcal{C}$. The following are equivalent

(i) $X \cong Y$

(ii) There is a family of bijections $\Phi_Z : \mathcal{C}(Z, X) \xrightarrow{\cong} \mathcal{C}(Z, Y)$ which is *natural in* Z, i.e. if $f : Z \to Z'$ is a morphism in $\mathcal{C}$, the following diagram commutes

$$\begin{array}{ccc}
\mathcal{C}(Z', X) & \xrightarrow{\Phi_{Z'}} & \mathcal{C}(Z', Y) \\
{\scriptstyle (-)\circ f}\downarrow & & \downarrow{\scriptstyle (-)\circ f} \\
\mathcal{C}(Z, X) & \xrightarrow{\Phi_Z} & \mathcal{C}(Z, Y)
\end{array}$$

(iii) There is a family of bijections $\Psi_Z : \mathcal{C}(X, Z) \xrightarrow{\cong} \mathcal{C}(Y, Z)$ which is *natural in* Z, i.e. if $f : Z \to Z'$ is a morphism in $\mathcal{C}$, the following diagram commutes

$$\begin{array}{ccc}
\mathcal{C}(X, Z) & \xrightarrow{\Psi_Z} & \mathcal{C}(Y, Z) \\
{\scriptstyle f\circ(-)}\downarrow & & \downarrow{\scriptstyle f\circ(-)} \\
\mathcal{C}(X, Z') & \xrightarrow{\Psi_{Z'}} & \mathcal{C}(Y, Z')
\end{array}$$

Proof. By proposition 2.32, a natural transformation between functors (in this case $\mathcal{C}(-, X)$ and $\mathcal{C}(-, Y)$) is an isomorphism if and only if it is a point-wise isomorphism (in this case, each component is a bijection between sets, since the target category is **Set**). Thus, the condition (ii) (resp. (iii)) is just stating that there is an isomorphism $\mathcal{C}(-, X) \cong \mathcal{C}(-, Y)$ (resp. $\mathcal{C}(X, -) \cong \mathcal{C}(Y, -)$ $\square$

Remark 7.6. This corollary can be read as saying that objects are determined up-to-isomorphism by the morphisms into or out of them. This is a manifestation of a core idea in category theory, that kinds of mathematical objects should be understood in the *context* of all the other mathematical objects of that kind. Two mathematical objects are isomorphic if and only if they play the same *role* in the appropriate category. For example, the role of the group $\mathbb{Z}$ in the category of abelian groups is that it represents the forgetful functor $U : \mathbf{Grp} \to \mathbf{Set}$, meaning that $\mathrm{Hom}(\mathbb{Z}, G) \cong U(G)$ for every abelian group G, and this bijection is natural in G. Any other abelian group H with this property is isomorphic to $\mathbb{Z}$, since $\mathrm{Hom}(\mathbb{Z}, -) \cong U \cong \mathrm{Hom}(H, -)$ and Corollary 7.5 implies they are isomorphic.
This is why the Yoneda Lemma is sometimes referred to as the *fundamental theorem of category theory.*

Example 7.7. Let k be an infinite field and denote by $\mathbf{Vect}_k^{fd}$ the category whose objects are finite dimensional vector spaces over k and whose arrows are k-linear maps. Recall that two objects in $\mathbf{Vect}_k^{fd}$ are isomorphic if and only if they have the same dimension over k. Now let V, V' be two objects of $\mathbf{Vect}_k^{fd}$ such that $0 < \mathrm{dim}_k V < \mathrm{dim}_k V'$. For any other object W of $\mathbf{Vect}_k^{fd}$, we have that there exists a bijection between $\mathbf{Vect}_k^{fd}(V, W)$ and $\mathbf{Vect}_k^{fd}(V', W)$. This is because if $\mathrm{dim}_k W = 0$, both these sets contain exactly one element, namely the linear map that maps everything to zero. If $\mathrm{dim}_k W > 0$, we know that $\mathbf{Vect}_k^{fd}(V, W)$ is in bijection with the set of $\mathrm{dim}_k V \times \mathrm{dim}_k W$ matrices over k, which is in bijection with $k^{(\mathrm{dim}_k V)(\mathrm{dim}_k)}$. But since k is infinite and the exponent is finite, we know from elementary set theory that this is in bijection with k. We can argue analogously that $\mathbf{Vect}_k^{fd}(V', W)$ is also in bijection with k. Recall that we assumed that $\dim_k V < \dim_k V'$, so in particular V is not isomorphic to V' in $\mathbf{Vect}_k^{fd}$, while their hom sets are isomorphic in the category of sets for every W of $\mathbf{Vect}_k^{fd}$. This shows that the naturality condition in Corollary 7.5 is actually necessary.

Example 7.8. Let V, W be $\mathbf{k}$-vector spaces and consider the following functor: $Bilin(V, W; -) : Vect_{\mathbf{k}} \to \mathbf{Set}$ which sends a vector space U to the set of bilinear maps from $V \times W$ to U. A linear map $f : U \to U'$ is sent to $f_* : Bilin(V, W; U) \to Bilin(V, W, U')$ which sends a bilinear map $g : V \times W \to U$ to the bilinear map $f \circ g : V \times W \to U'$ in $Bilin(V, W; U')$

A representation of the functor $Bilin(V, W; -)$ is a $\mathbf{k}$ vector space $V \otimes_{\mathbf{k}} W$, the tensor product of V and W. So the tensor product is defined by an isomorphism

$$Vect_k(V \otimes_k W, Z) \cong Bilin(V, W; Z) \tag{2}$$

where the isomorphism is natural in Z.

Let $\otimes : V \times W \to V \otimes_k W$ be the image of the identity $1_V \otimes_k W$ under the isomorphism in (2). The natural isomorphism above identifies a bilinear map $f : V \times W \to Z$, with a linear map $f^\# : V \otimes_k W \to Z$. By the naturality condition induced by $f^\# : V \otimes_k W \to Z$, we have the following commutative diagram

$$\begin{tikzcd}
Vect_k(V \otimes_k W, V \otimes_k W) \arrow[r, "\cong"] \arrow[d, "(f^\#)_*"'] & Bilin(V, W; V \otimes_k W) \arrow[d, "(f^\#)_*"] \\
Vect_k(V \otimes_k W, Z) \arrow[r, "\cong"'] & Bilin(V, W; Z)
\end{tikzcd}$$

Tracing the identity $1_V \otimes_k W$ gives that the bilinear map f factors uniquely through the bilinear map $\otimes$ along the linear map $f^\#$.

$$\begin{tikzcd}
V \times W \arrow[r, "f"] \arrow[d, "\otimes"'] & Z \\
V \otimes_k W \arrow[ur, dashed, "f^\#"'] &
\end{tikzcd}$$

The defining universal property of the tensor product of V and W also supplies as with a method for its construction. Suppose $V \otimes_k W$ exists and consider its quotient by the vector space spanned by the image of the bilinear map $- \otimes -$. We have that the quotient map $V \otimes_k W \to V \otimes_k W / < v \otimes w >$ composed with the linear map $- \otimes -$ gives the zero map. But the zero map $V \otimes_k W \to V \otimes_k W / < v \otimes w >$ also has this property, and so, by the universal property of tensor product, the zero map and the quotient map should coincide. Since the quotient map is surjective, we get that $V \otimes_k W$ is isomorphic to the span of the vectors $v \otimes w$ modulo the bilinearity condition on $- \otimes -$.

Proposition 7.9. For any $\mathbf{k}$-vector space V, we have $V \otimes_{\mathbf{k}} \mathbf{k} \cong V$ where $\mathbf{k}$ is viewed as a $\mathbf{k}$-vector space.

Proof. There exists an isomorphism between $Bilin(V, \mathbf{k}; Z)$ and $Vect_{\mathbf{k}}(V, Z)$ which is natural in Z. The isomorphism sends a $\mathbf{k}$-linear map $f : V \to Z$ to the bi-linear map $f^\# : V \times \mathbf{k} \to Z$ defined by $f^\#(v, \alpha) = \alpha f(v)$. Thus, we have

$$Vect_{\mathbf{k}}(V \otimes_{\mathbf{k}} \mathbf{k}, Z) \cong Bilin(V, \mathbf{k}; Z) \cong Vect_{\mathbf{k}}(V, Z)$$

and the isomorphisms are natural in Z. So by Corollary 7.4, we get that $V \otimes_{\mathbf{k}} \mathbf{k} \cong V$. $\square$

7.2 The Cayley Theorem from the Yoneda Lemma

We saw earlier that we can regard a group G as a small one-object category BG where each arrow is an isomorphism. The Cayley Theorem says that G is isomorphic to a permutation group on G, i.e., a subgroup of the symmetric group on the underlying set of G. For a set X, the *symmetric group* on X, written S_X, is the set of all bijective functions $f : X \to X$, with binary operation given by composition.

Theorem 7.10 (Cayley)**.** Let G be a group. Then G is isomorphic to a subgroup of S_G.

Proof. Let BG be the category corresponding to G. Then BG has only object $*$ and the set of arrows is equal to $\mathrm{Hom}_{BG}(*.*) = G$. The Contravariant Yoneda Lemma now gives us a bijection

$$\Phi : \mathrm{Nat}(h_*, h_*) \to h_*(*) = \mathrm{Hom}_{BG}(*, *) = G$$

given by $\Phi(\eta) = \eta_*(\mathrm{Id}_*)$. Let now $x \in G$. Then, by the above, there is corresponding natural tranformation $\eta_x \in \mathrm{Nat}(h_*, h_*)$. As BG has only a single object,η_x consists of a single map $f_x = (\eta_x)_* : h_*(*) \to h_*(*)$. In particular, for $x, y \in G$ with $x \neq y$ we have that $f_x \neq f_y$ as otherwise $(\eta_x)_* = (\eta_y)_*$,and since $*$ is the only object of BG,consequently $\eta_x = \eta_y$, which contradicts Φ being a bijection. We now have an injection $\varphi : G \to S_G$ given by $\varphi(x) = f_x$. By naturality of $\eta_x : h_* \Rightarrow h_*$ we get that

$$\begin{tikzcd}
h_*(*) \arrow[r, "f_x"'] \arrow[d, "h_*(\overline{y})"'] & h_*(*) \arrow[d, "h_*(\overline{y})"] \\
h_*(*) \arrow[r, "f_x"] & h_*(*)
\end{tikzcd}$$

commutes for each $y \in \mathrm{Hom}_{BG}(*, *) = G$. That is, for $z \in \mathrm{Hom}_{BG}(*, *) = G$, we have

$$f_x(z) \circ y = h_*(y)(f_x(z)) = f_x(h_*(y)(z)) = f_x(z \circ y).$$

If we let z be the identity $e \in G$, we have

$$f_x(y) = f_x(e \circ y) = f_x(e) \circ y = (\eta_x)_*(e) \circ y = \Phi(\eta_x) \circ y = x \circ y = x \cdot y.$$

Let now $x, y \in G$ and also $w \in G$. Then we find

$$(f_x \circ f_y)(w) = f_x(f_y(w)) = f_x(y \cdot w) = x \cdot (y \cdot w) = (x \cdot y) \cdot w = f_{x \cdot y}(w).$$

We see that $\phi : G \to S_G : x \mapsto f_x$ is an injective group homomorphism. In particular we have now that G is isomorphic to a subgroup of S_G. $\square$

7.3 Characterization of polynomials on rings

In this section we consider the category **Ring** of rings (with identity). We adapt the final part of [3]. For some ring R, we denote by $R[X]$ the ring of formal polynomials with coefficients in R. Then for $P = r_0X^0 + r_1X^1 + ... + r_nX^n$ some element in $\mathbb{Z}[X]$, we have for each ring R an *interpretation* P_R, that is, a function $R \to R$, not necessarily a homomorphism, which sends $x \in R$ to $r_0 \cdot 1 + r_1 \cdot x + r_2 \cdot x^2 + ... + r_n \cdot x^n$ where $r_0 \cdot x$ denotes adding x to itself r_i times, so for example $0 \cdot x = 0$ and $3 \cdot x = x + x + x$. We first characterize the homomorphisms from $\mathbb{Z}[X]$ to some ring R. We let $U : \mathbf{Ring} \to \mathbf{Set}$ be the forgetful functor that sends a ring R to its underlying set and a ring homomorphism ϕ to its underlying function. We also denote by $\pi_{R,r} : \mathbb{Z}[X] \to R$ the homomorphism that sends an element $P \in \mathbb{Z}[X]$ to the evaluation of P_R at the element $r \in R$.

Lemma 7.11. There is a natural isomorphism $\pi : U \Rightarrow h^{\mathbb{Z}[X]}$ given by $\pi_R(r) = \pi_{R,r}$. The inverse π^{-1} is given by $\pi_R^{-1}(\varphi) = \varphi(X)$.

Proof. Let for R a ring $\pi_R : R \to h^{\mathbb{Z}[X]}(R) = \mathrm{Hom}_{\mathbf{Ring}}(\mathbb{Z}[X], R)$ be a function defined as above. We first show that π_R is a bijection. Let $\varphi : \mathbb{Z} \to R$ be a ring homomorphism. Let $r = \varphi(X)$. Then we have for $P = r_0X^0 + r_1X^1 + ... + r_nX^n$ in $\mathbb{Z}$ that

$$\begin{aligned}
\varphi(P) &= \varphi(r_0X^0 + r_1X^1 + ... + r_nX^n) \\
&= \varphi(r_0 \cdot x^0) + ... + \varphi(r_n \cdot X^n) \\
&= r_0 \cdot \varphi(X^0) + ... + r_n \cdot \varphi(X^n) \\
&= r_0 \cdot \varphi(1) + r_1 \cdot (\varphi(X))^1 + ... + r_n \cdot (\varphi(X))^n \\
&= r_0 \cdot r^0 + r_1 \cdot r^1 + ... + r_n \cdot r^n \\
&= P_R(r) \\
&= \pi_{R,r}(P).
\end{aligned}$$

We see $\varphi = \pi_{R,r} = \pi_R(r)$. This shows that π_R is surjective. Let now $r, s \in R$ with $\pi_R(r) = \pi_R(s)$. Then we have

$$r = (X)_R(r) = \pi_{R,r}(X) = (\pi_R(r))(X) = (\pi_R(s))(X) = \pi_{R,s}(X) = (X)_R(s) = s$$

which shows π_R is injective, and consequently, bijective. We now show that the π_R form a natural transformation.

Let $\varphi : R \to S$ be a ring homomorphism and let $r \in R, P = r_0X^0 + r_1X^1 + ... + r_nX^n \in \mathbb{Z}[X]$. Then we have

$$\begin{aligned}
(\pi_S \circ U(\varphi))(r)(P) &= \pi_S(\varphi(r))(P) \\
&= \pi_{S,\varphi(r)}(P) \\
&= P_S(\varphi(r)) \\
&= r_0 \cdot (\varphi(r))^0 + ... + r_n \cdot (\varphi(r))^n \\
&= r_0 \cdot \varphi(r^0) + ... + r_n \cdot \varphi(r^n) \\
&= \varphi(r^0 \cdot r^0 + ... + r_n \cdot r^n) \\
&= \varphi(P_R(r)) \\
&= \varphi(\pi_R(r)(P)) \\
&= (\varphi \circ \pi_R(r))(P) \\
&= (h^{\mathbb{Z}[X]}(\varphi) \circ \pi_R)(r)(P).
\end{aligned}$$

This proves naturality. We conclude that we have a natural isomorphism $\pi : U \Rightarrow h^{\mathbb{Z}[X]}$. The form of the inverse is clear from the proof of surjectivity. □

We now prove the following theorem using the Covariant Yoneda Lemma.

Theorem 7.12. There is a 1-1 correspondence between (necessarily class sized) collections $F = (F_R | R \text{ a ring})$ of functions such that for each ring homomorphism $\phi : R \to S$ we have $\phi \circ F_R = F_S \circ \phi$ as functions and elements P of $\mathbb{Z}[X]$. In particular if P corresponds to the collection F, then it is the unique element of $\mathbb{Z}[X]$ such that $P_R = F_R$ for every ring R.

Proof. Let F be a collection as above. Then, for each ring homomorphism $\phi : R \to S$, $\phi \circ F_R = F_S \circ \phi$ as functions, i.e., the follow diagram commutes in **Set**.

$$\begin{tikzcd}
R \arrow[r, "F_R"] \arrow[d, "U(\phi)=\phi"'] & R \arrow[d, "\phi=U(\phi)"] \\
S \arrow[r, "F_S"'] & S
\end{tikzcd}$$

This says precisely that the collection F is a natural transformation $U \Rightarrow U$.

Let now π be as in the preceding lemma. We define a function $\Pi : \text{Nat}(h^{\mathbb{Z}[X]}, U) \to \text{Nat}(U, U)$ by $\Pi(F) = \eta \circ \pi$. Let η, ε be natural transformations $U \Rightarrow U$ with $\Pi(\eta) = \Pi(\varepsilon)$. Then we have

$$\eta = \eta \circ 1_{h^{\mathbb{Z}[X]}} = \eta \circ \pi \circ \pi^{-1} = \Pi(\eta) \circ \pi^{-1} = \Pi(\varepsilon) \circ \pi^{-1} = \varepsilon \circ \pi \circ \pi^{-1} = \varepsilon \circ 1_{h^{\mathbb{Z}[X]}} = \varepsilon.$$

This shows that Π is injective. Let now F be a natural transformation $U \Rightarrow U$. Then we have $F \circ \pi^{-1} \in \text{Nat}(h^{\mathbb{Z}[X]}, U)$ and

$$\Pi(F \circ \pi^{-1}) = F \circ \pi^{-1} \circ \pi = F.$$

We conclude that Π is a bijection and Π^{-1} is given by $\Pi^{-1}(F) = F \circ \pi^{-1}$.

The Covariant Yoneda Lemma now gives us another bijection $\Phi : \mathrm{Nat}(h^{\mathbb{Z}[X]}, U) \to U(\mathbb{Z}[X]) = \mathbb{Z}[X]$, given by $\Phi(\eta) = \eta_{\mathbb{Z}[X]}(1_{\mathbb{Z}[X]})$. Note now that $\Phi \circ \Pi^{-1}$ is a bijection $\mathrm{Nat}(U, U) \to \mathbb{Z}[X]$. This gives us our 1-1 correspondence. Let now $F : U \Rightarrow U$. Then we have

$$(\Phi \circ \Pi^{-1})(F) = \Phi(F \circ \pi^{-1}) = (F \circ \pi^{-1})_{\mathbb{Z}[X]}(1_{\mathbb{Z}[X]}) = F_{\mathbb{Z}[X]}(\pi^{-1}_{\mathbb{Z}[X]}(1_{\mathbb{Z}[X]})) = F_{\mathbb{Z}[X]}(1_{\mathbb{Z}[X]}(X))$$

For a ring R and $r \in R$ we now get

$$((\Phi \circ \Pi^{-1})(F))_R(r) = (F_{\mathbb{Z}[X]}(1_{\mathbb{Z}[X]}(X)))_R(r) = (F_{\mathbb{Z}[X]}(X))_R(r) = \pi_{R,r}(F_{\mathbb{Z}[X]}(X)).$$

By naturality of F, the above becomes

$$((\Phi \circ \Pi^{-1})(F))_R(r) = F_R(\pi_{R,r}(X)) = F_R(r).$$

This shows that F corresponds to an element P of $\mathbb{Z}[X]$ such that $P_R = F_R$ for each ring R. By the 1-1 correspondence, P is unique with this property. This completes the proof. $\square$

8 Constructions in the category of presheaves

In this section, we explore the structure of the category of presheaves, see how they naturally solve representability problems such as the exponential object, and then explore an example construction of exponentials in the presheaf category of graphs.

8.1 Working with presheaves

Many constructions in category theory, such as limits, colimits, adjunctions etc, involve reasoning about diagrams. Since presheaves are set-valued, the first approach to reasoning about presheaves, for example, verifying two natural transformations agree, is to calculate that they agree pointwise (at every $C \in \mathcal{C}$) by explicit equations (with many subscripts!). We demonstrate how the naturality of the map $\Phi_{C,F}$ in C and F allows us to reason at the level of objects and morphisms of the presheaf category $\widehat{\mathcal{C}}$.

Notation 8.1. In this section, we adopt the notation $\mathbf{y}$ for the covariant functor $h_{(-)} : \mathcal{C} \to \widehat{\mathcal{C}}$ sending C to $\mathrm{Hom}(-, C)$. Recall that this is the Yoneda embedding from Section 7.

First we provide a way to externalise the internal structure of a presheaf $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$. Naturality of $\Phi_{C,F}$ in C says that $\Phi_{(-),F}$ is a natural isomorphism between $\mathrm{Nat}(\mathbf{y}(-), F) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ and $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$, i.e. these are the same presheaf up to the isomorphism given by $\Phi_{(-),F}$. So not only can we identify elements of $x \in F(C)$ with natural transformations $\mathbf{y}(C) \xrightarrow{x} F$, we can also identify the restriction $F(f) : F(C) \to F(D)$ (for a morphism $f : D \to C$) with pre-composition by $\mathbf{y}(f)$ (this is the action of $\mathrm{Nat}(\mathbf{y}(-), F)$ on morphisms). This is expressed by the following commuting diagram.

$$\begin{tikzcd}
\alpha \arrow[d, mapsto] & \in & \mathrm{Nat}(\mathbf{y}(C), F) \arrow[r, "\Phi_{C,F}"] \arrow[d, "(-)\circ\mathbf{y}(f)"'] & F(C) \arrow[d, "F(f)"] & \ni & x \arrow[d, mapsto] \\
\alpha \circ \mathbf{y}(f) & \in & \mathrm{Nat}(\mathbf{y}(D), F) \arrow[r, "\Phi_{D,F}"] & F(D) & \ni & F(f)(x)
\end{tikzcd}$$

We take this isomorphism seriously by writing x for α, and since $\mathbf{y}$ is fully-faithful, we write f for $\mathbf{y}(f)$. Now instead of saying $F(f)(z) = x$ we can say this diagram commutes

$$\begin{tikzcd}
\mathbf{y}(C) \arrow[r, "x"] & F \\
\mathbf{y}(D) \arrow[u, "f"] \arrow[ur, "z"'] &
\end{tikzcd}$$

The correspondence can be taken further by unfolding what it means to say that $\Phi_{C,F}$ is natural in F. This means that for fixed C, $\Phi_{C,(-)}$ is a natural isomorphism between the functors $\mathrm{Nat}(\mathbf{y}(C), -) : \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}} \to \mathbf{Set}$ and $\mathrm{ev}(\mathrm{C}, -) : \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}} \to \mathbf{Set}$ (the latter functor sends the presheaf F to $F(C)$). Given a natural transformation $\eta : F \Rightarrow G$ between presheaves, these functors evaluated at η are the following functions (respectively).

$$\eta \circ (-) : \mathrm{Nat}(\mathbf{y}(C), F) \to \mathrm{Nat}(\mathbf{y}(C), G)$$

$$\eta_C : F(C) \to G(C)$$

Thus we have a commuting diagram

$$\begin{array}{ccccccc}
\alpha & \in & \mathrm{Nat}(\mathbf{y}(C), F) & \xrightarrow{\Phi_{C,F}} & F(C) & \ni & x \\
\big\downarrow & & \big\downarrow{\scriptstyle \eta\circ(-)} & & \big\downarrow{\scriptstyle \eta_C} & & \big\downarrow \\
\eta\alpha & \in & \mathrm{Nat}(\mathbf{y}(C), G) & \xrightarrow{\Phi_{C,G}} & G(C) & \ni & \eta_C(x)
\end{array}$$

and under this correspondence, we can replace the statement $\eta_C(x) = z$ by commutativity of this diagram

$$\begin{tikzcd}
\mathbf{y}(C) \arrow[r, "x"] \arrow[dr, "z"'] & F \arrow[d, "\eta"] \\
& G
\end{tikzcd}$$

In summary, the *internal* data of a presheaf can be described *externally* via morphisms $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ from the representable presheaves, and the equality of elements of a presheaf after restriction or applying a natural transformation, can be rephrased as commutativity of diagrams in $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$.

8.2 Exponential Objects

The concept of representability is widely re-usable and provides a framework for defining many kinds of objects in a category. One such example is that of an *exponential object.*

Definition 8.2. Let $\mathcal{C}$ be a category with binary products and X, Y objects. Consider the functor $\mathcal{C}(X \times (-), Y) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$, which sends an object A to the hom-set $\mathcal{C}(X \times A, Y)$, and a morphism $f : A \to B$ to the operation of precomposition by $(1_X \times f) : X \times A \to X \times B$. An object is an *exponential* of X and Y if and only if it is a representative of the functor $\mathcal{C}((-) \times X, Y)$. We denote such an object Y^X.

Example 8.3. In the category of sets, $Y^X := \{f : X \to Y\}$ represents the functor $\mathbf{Set}((-) \times X, Y)$.

Proof. We must provide a natural isomorphism $\mathbf{Set}(-, Y^X) \xrightarrow{\cong} \mathbf{Set}((-) \times X, Y)$. By the Yoneda Lemma, natural transformations of this form are in bijection with elements of $\mathbf{Set}(Y^X \times X, Y)$. One such element is the evaluation function $\mathrm{ev} : Y^X \times X \to Y$, defined $\mathrm{ev}(f, x) = f(x)$. The corresponding natural transformation is defined at a set Z as follows

$$\mathbf{Set}(Z, Y^X) \xrightarrow{\alpha_Z} \mathbf{Set}(Z \times X, Y)$$

$$g \longmapsto \mathrm{ev} \circ (g \times 1_X)$$

(note that $\alpha_{Y^Z}(1_{Y^X})$ recovers our chosen element ev, since $\mathrm{ev} \circ (1_{Y^X} \times 1_X) = \mathrm{ev}$. To see that this is a bijection, we define an inverse

$$\mathbf{Set}(Z \times X, Y) \xrightarrow{\alpha_Z^{-1}} \mathbf{Set}(Z, Y^X)$$
$$h \longmapsto \alpha_Z^{-1}(h)$$

where $\alpha_Z^{-1}(h)$ is defined $\alpha_Z^{-1}(h)(z)(x) = g(z,x)$. Now we calculate

$$\begin{aligned}
\alpha_Z(\alpha_Z^{-1}(h)) &= (\mathrm{ev} \circ (\alpha_Z^{-1}(h) \times 1_X))(z,x) \\
&= \mathrm{ev}(\alpha_Z^{-1}(h)(z), x) \\
&= \alpha_Z^{-1}(h)(z)(x) \\
&= h(z,x)
\end{aligned}$$

So the composition $\mathbf{Set}(Z \times X, Y) \to \mathbf{Set}(Z, Y^X) \to \mathbf{Set}(Z \times X, Y)$ is the identity. For the other composition, we compare a given function $g : Z \to Y^X$ to $\alpha_Z^{-1}(\alpha_Z(g))$. We find that

$$\begin{aligned}
\alpha_Z^{-1}(\alpha_Z(g)) &= \alpha_Z^{-1}(\mathrm{ev} \circ (g \times 1_X))(z)(x) \\
&= (\mathrm{ev} \circ (g \times 1_X))(z,x) \\
&= \mathrm{ev}(g(z), x) \\
&= g(z)(x)
\end{aligned}$$

so we recover the behaviour of g. If $\alpha_Z(g) = h$ we say that g is the *exponential transpose* of h. In a computer science context, g is referred to as the *curried* form of the function h. $\square$

8.3 Exponentials in presheaf categories

Not all categories have exponential objects, so for a given category, one must first determine an object E which is a solution to the representation problem:

$$\mathrm{Hom}(-, E) \cong \mathrm{Hom}((-) \times X, Y).$$

The situation is inverted in a presheaf category $\widehat{\mathbf{y}}$, where the representation problem actually provides a definition of the presheaf that solves it.
Suppose X and Y are presheaves, and we want to find a presheaf that represents the functor $\widehat{\mathcal{C}}((-) \times X, Y) : (\widehat{\mathcal{C}})^{\mathrm{op}} \to \mathbf{Set}$. This is a presheaf on$\widehat{\ }$, but we can obtain a presheaf on $\mathcal{C}$ by precomposing with the Yoneda embedding

$$\mathcal{C}^{\mathrm{op}} \xhookrightarrow{\mathbf{y}} (\widehat{\mathcal{C}})^{\mathrm{op}} \xrightarrow{\widehat{\mathcal{C}}((-)\times X, Y)} \mathbf{Set}$$

This is the presheaf of sets $\{\widehat{\mathcal{C}}(\mathbf{y}(C) \times X, Y)\}_{C \in \mathcal{C}}$ with restriction operation given by $e \cdot f = e \circ (\mathbf{y}(f) \times 1_X)$, for $e \in \widehat{\mathcal{C}}(\mathbf{y}(C) \times X, Y)$ and $f : D \to C$.

Proposition 8.4. The presheaf $Y^X := \{\widehat{\mathcal{C}}(\mathbf{y}(C) \times X, Y)\}_{C \in \mathcal{C}}$ is a representative of the functor $\widehat{\mathcal{C}}((-) \times X, Y)$, and is therefore an exponential of X and Y

The proof of this fact requires some additional facts about presheaves and properties of the functor $\widehat{\mathcal{C}}((-) \times X, Y)$, but first note that the natural isomorphism $\alpha = (\alpha_Z : \widehat{\mathcal{C}}(Z, Y^X) \to \widehat{\mathcal{C}}(Z \times X, Y))_{Z \in \widehat{\mathcal{C}}}$ is automatically given by the Yoneda Lemma when Z is representable, since the Yoneda Lemma says $\widehat{\mathcal{C}}(\mathbf{y}(C), Y^X) \cong Y^X(C)$, which we defined to be $\widehat{\mathcal{C}}(\mathbf{y}(C) \times X, Y))_Z$. To extend this isomorphism to arbitrary presheaves we need the following important facts.

Lemma 8.5. Any presheaf $X : \mathcal{C}^{\text{op}} \to \mathbf{Set}$ can be written as a colimit of representable presheaves, i.e., there is a diagram $F : \mathcal{I} \to \mathcal{C}$ such that X is the colimit of the composite diagram $\mathbf{y} \circ F : \mathcal{I} \to \widehat{\mathcal{C}}$.

Lemma 8.6. Let $\mathcal{D}$ be any category, and D an object of $\mathcal{D}$. The hom-functor $\mathcal{D}(-, D) : \mathcal{D}^{\text{op}} \to \mathbf{Set}$ preserves limits, and if $\mathcal{D}$ has binary products, the functor $(-) \times D : \mathcal{D} \to \mathcal{D}$ preserves colimits.

We defined what it means for functors to preserve limits and colimits in the Basics section. Proof of these lemmas can be found in [4]. We can now complete the proof.

Proof. Suppose Z is a colimit of the diagram $\mathbf{y} \circ F : \mathcal{I} \to \widehat{C}$, which consists of representable presheaves $\mathbf{y}(C_i)$ for $i \in \mathcal{I}$. Now using that $\widehat{\mathcal{C}}(-, Y^X) : \mathcal{C}^{\text{op}} \to \mathbf{Set}$ preserves limits, we have a natural isomorphism $\widehat{\mathcal{C}}(Z, Y^X) \cong \widehat{\mathcal{C}}(\text{colim}_I\, \mathbf{y}(C_i), Y^X) \cong \lim_I \widehat{\mathcal{C}}(\mathbf{y}(C_i), Y^X)$. Here we have used that a limit in $(\widehat{\mathcal{C}})^{\text{op}}$ is precisely a colimit in $\widehat{\mathcal{C}}$ by the duality in the definitions of limit and colimit. This is why the colimit inside the Hom-functor $\widehat{\mathcal{C}}(-, Y^X)$ becomes a limit outside of it.
We continue this chain of natural isomorphisms by noticing we have reduced to the representable case, after which we reach $\widehat{\mathcal{C}}(Z \times X, Y)$ by repeatedly applying preservation of limits/colimits.

$$\begin{aligned}
\lim_I \widehat{\mathcal{C}}(\mathbf{y}(C_i), Y^X) &\cong \lim_I \widehat{\mathcal{C}}(\mathbf{y}(C_i) \times X, Y) \\
&\cong \widehat{\mathcal{C}}(\operatorname*{colim}_I (\mathbf{y}(C_i) \times X), Y) \\
&\cong \widehat{\mathcal{C}}(\operatorname*{colim}_I \mathbf{y}(C_i) \times X, Y) \\
&\cong \widehat{\mathcal{C}}(Z \times X, Y)
\end{aligned}$$

□

8.4 Exponentials in the category of Graphs

In section 5.4, we saw that the category of directed multigraphs can be realised as a presheaf category. We apply the results of the previous section to obtain from graphs G and H the appropriate definition for the exponential graph H^G.

Notation 8.7. We write $\mathbf{Graph} := [\mathbf{2}^{\text{op}}, \mathbf{Set}]$ for the category of presheaves on the category $\mathbf{2}$ consisting of two objects $0, 1$ and two non-identity morphisms $s, t : 0 \to 1$. A directed multi-graph G, regarded as a presheaf $G \in \mathbf{Graph}$, consists of a vertex set $G_0 := G(0) \in \mathbf{Set}$ and an edge set $G_1 = G(1) \in \mathbf{Set}$. We write $\mathfrak{s}, \mathfrak{t} : G_1 \to G_0$ for the two restriction functions given by the non-identity morphisms $s, t : 0 \to 1$ in the category $\mathbf{2}$, which send an edge to its source and target vertex respectively. We write u for the unique vertex of the representable graph $\mathbf{y}(0) := \text{Hom}_{\mathbf{2}}(-, 0)$ and we label the vertices and edges of $\mathbf{y}(1) := \text{Hom}_{\mathbf{2}}(-, 1)$ as below:

$$v_0 \xrightarrow{e} v_1$$

We describe products of graphs.

Definition 8.8. Let G, H be graphs. The product graph $(G \times H)$ has vertex set $(G \times H)_0 := G_0 \times H_0$ and edge set $(G \times H)_1 = G_1 \times H_1$. The source and target functions are given by $\mathfrak{s}(e_1, e_2) = (\mathfrak{s}e_1, \mathfrak{s}e_2)$, and $\mathfrak{t}(e_1, e_2) = (\mathfrak{t}e_1, \mathfrak{t}e_2)$

The correctness of this definition (as a limit in **Graph**) is a special case of the fact that functor categories of the form $[\mathcal{C}, \mathcal{D}]$ have whatever limits and colimits that exist in the target category $\mathcal{D}$, and are computed "pointwise". For presheaves, this means that for a diagram $F : \mathcal{I} \to \mathcal{C}^{\mathrm{op}}$, $\lim_I F(i)$ is the presheaf of sets, defined at C as $\lim_I F(i)(C)$. For a proof of this fact see Section 6.2 of [4].

Lemma 8.9. A morphism of graphs $f : G \to H$ is a pair $f_0 : G_0 \to H_0$ (the vertex map) and $f_1 : G_1 \to H_1$ (the edge map) such that $f_0(\mathfrak{s}e) = \mathfrak{s}f_1(e)$ and $f_0(\mathfrak{t}(e)) = \mathfrak{t}f_1(e)$.

Proof. This is a restatement of the data and conditions of a natural transformation between G and H, as given in section 5.4. $\square$

Definition 8.10. Let G, H be graphs. The *exponential graph* H^G has vertex set $(H^G)_0 := \mathbf{Set}(G_0, H_0)$, the set of functions between the vertex sets and edge set $(H^G)_1 := \mathbf{Graph}(\mathbf{y}(1) \times G, H)$. Let $f : \mathbf{y}(1) \times G \to H$ be morphisms of graphs. The source map sends f to the function on vertices $\mathfrak{s}(f) \in \mathbf{Set}(G_0, H_0)$ given by $\mathfrak{s}(f)(v) = f_0(v_0, v)$. Similarly $\mathfrak{t}(f)(v) = f_0(v_1, v)$.

Theorem 8.11. The graph H^G is an exponential object for G and H in the sense of definition 8.2, and thus there is a natural bijection $\mathbf{Graph}(K, H^G) \cong \mathbf{Graph}(K \times G, H)$ for all graphs K.

Proof. We show that the exponential H^G as defined is equivalent to the construction given for a general presheaf in the previous section, which would define $(H^G)_0 = \mathbf{Graph}(\mathbf{y}(0) \times G, H)$ and $(H^G)_1 = \mathbf{Graph}(\mathbf{y}(1) \times G, H)$. Observe that the graph $\mathbf{y}(0)$ has no edges, so $(\mathbf{y}(0) \times G)_1 = (y_0)_1 \times G_1 = \emptyset \times G_1 = \emptyset$, and therefore a morphism of graphs $\mathbf{y}(0) \times G \to H$ is just the data of a function on vertices $G_0 \to H_0$.
From the presheaf-exponential construction, the source map $\mathbf{Graph}(\mathbf{y}(1) \times G, H) \to \mathbf{Graph}(\mathbf{y}(0) \times G, H)$ sends f to the graph morphism $f \circ (\mathbf{y}(s) \times 1_G)$. Recall that u is our relabelling of the identity morphism $\mathrm{Id}_0 : 0 \to 0$ in $(\mathbf{y}(0))(0) = \mathbf{2}(0, 0)$, so for a vertex pair $(\mathrm{Id}_0, v) \in (\mathbf{y}(0) \times G)_0$,

$$(f \circ (\mathbf{y}(s) \times 1_G))_0(\mathrm{Id}_0, v) = f_0(s, v)$$

but $s \in (\mathbf{y}(1))_0$ is the vertex of $\mathbf{y}(1)$ we called v_0. So the corresponding function $G_0 \to H_0$ is indeed given by $\mathfrak{s}(f)(v) = f_0(v_0, v)$. The same proof shows the correctness of the target map, with t in place of s. $\square$

Example 8.12. We give an example to illustrate the edges of the exponential graph H^G. Take G to be a graph with a single edge c and vertices a, b. Then $G \times \mathbf{y}(1)$ has four vertices $(a, v_0), (b, v_0), (a, v_1), (1, v_1)$ and a single edge (c, e) from (a, v_0) to (b, v_1). Then an edge $f \in \mathbf{Graph}(G \times \mathbf{y}(1), H)$ can be visualised as a subgraph of H of this shape.

$$\begin{tikzcd}
(G & \times & \mathbf{y}(1) &) \arrow[r, "f"] & H & \\
a \arrow[dd, "c"] & & & & (a,v_0) \arrow[ddr, "{(e,c)}"] & (a,v_1) \\
& v_0 \arrow[rr, "e"] & & v_1 & & \\
b & & & & (b,v_0) & (b,v_1)
\end{tikzcd}$$

Although this is the correct notion of exponential for **Graph** as presented, there is evidently a lack of structure in this example. For this choice of G, graph morphisms $G \to H$ correspond to edges in H, so we would expect edges in the exponential H^G to give a notion of edges "between" edges, however there is no constraint on the choices of (a, v_1) and (b, v_0), and only a single H-edge that relates the four vertices.

The situation can be improved by modifying the underlying category $\mathbf{2}$ to include an extra non-identity morphism $d : 1 \to 0$, with composition defined $ds = dt = \mathrm{Id}_0$.

$$\begin{tikzcd}
0 \arrow[r, "d"] & 1 \arrow[l, bend right, "s"'] \arrow[l, bend left, "t"]
\end{tikzcd}$$

This has the effect of adding "degenerate edges" to graphs such that for every vertex v, there is a new edge $\mathfrak{d}(v)$ with $\mathfrak{s}(\mathfrak{d}(v)) = \mathfrak{t}(\mathfrak{d}(v)) = v$. Here we use $\mathfrak{d} : G_0 \to G_1$ for restriction along the morphism d. Importantly $\mathbf{y}(0)$ now has a single vertex u *and* a single edge $\mathfrak{d}(u)$ from u to u, so $G \times \mathbf{y}(0) \cong G$, rather than being G with no edges. Therefore the vertices of the exponential are in bijection with graph morphisms $f : G \to H$ (not just maps of vertices). With naturality one can show that $f_1(\mathfrak{d}(v)) = \mathfrak{d}(f_0(v)$, and thus including degenerate edges does not fundamentally change the collection of graph morphisms from G to H.

We repeat the previous example for this modified presheaf category of graphs with degenerates.

Example 8.13. Let G to be a graph with degenerates, with vertices a, b and edges $\mathfrak{d}(a), \mathfrak{d}(b), c$. Then $G \times \mathbf{y}(1)$ has four vertices $(a, v_0), (b, v_0), (a, v_1), (1, v_1)$ and (non-degenerate) edges $(c, \mathfrak{d}v_0), (c, \mathfrak{d}v_1), (\mathfrak{d}a, e), (\mathfrak{d}b, e), (c, e)$. Then an edge $f \in \mathbf{Graph}(G \times \mathbf{y}(1), H)$ can be visualised as a subgraph of H of this shape.

$$\begin{tikzcd}
(G & \times & \mathbf{y}(1) &) \arrow[r, "f"] & H & \\
a \arrow[dd, "c"] \arrow[loop above] & & & & (a,v_0) \arrow[loop above] \arrow[r, "{(\mathfrak{d}a,e)}"] \arrow[dd, "{(c,\mathfrak{d}v_0)}"'] \arrow[ddr, "{(e,c)}"] & (a,v_1) \arrow[loop above] \arrow[dd, "{(c,\mathfrak{d}v_1)}"] \\
& v_0 \arrow[loop above] \arrow[rr, "e"] & & v_1 \arrow[loop above] & & \\
b \arrow[loop below] & & & & (b,v_0) \arrow[loop below] \arrow[r, "{(\mathfrak{d}b,e)}"] & (b,v_1) \arrow[loop below]
\end{tikzcd}$$

It is evident that this a stronger notion of an edge-between-edges. Observe that $\mathfrak{s}(f)$ is the left vertical edge of the square, and $\mathfrak{t}(f)$ is the right vertical edge.

Remark 8.14. This modification of the underlying category can be continued to add "higher-dimensional" sets to the presheaf category. For example, we can add a third object 2 to introduce the set G_2 of triangles, along with maps between 1 and 2 to relate each triangle to its three directed edges (much like edges are related to their vertices). Then a fourth for tetrahedrons, etc... The full realisation of this idea to arbitrary dimensions is given by using the *simplex category* Δ, with finite ordinals $[n] := \{0, 1, 2, \ldots, n\}$ as objects and monotone-maps (non-decreasing) as morphisms. The presheaf category on Δ is called the category of *simplicial sets*, **sSet**, is a discrete/categorical setting for homotopy theory. It is strongly related to the homotopy theory of CW-complexes, in fact, the notion of edge between graph-morphisms in the exponential graph H^G as a morphism $f : G \times \mathbf{y}(1) \to H$ is closely related to the topological notion of homotopy between continuous maps of 1-dimensional CW-complexes (this correspondence is called the *geometric realisation*). For more about simplicial sets, see [5].

9 Abelian Categories and Mitchell's Embedding Theorem

In this section we will give a short introduction to the theory of abelian categories and showcase yet another application of the Yoneda Lemma and the Yoneda Embedding. This will culminate in a sketch of the proof of the *Mitchell Embedding Theorem*. This theorem roughly states that every abelian category can be fully faithfully embedded into the category R-**mod**, where R-**mod** is the category of (left) R-modules over a certain ring R. To do this we will prove a slightly altered version of the Yoneda Lemma, which we will call the *Additive Yoneda Lemma*. We are going to define what an abelian category is in multiple steps and we encourage the reader to keep prototypical examples in mind throughout this process. But what are examples of abelian categories? An informal way of defining an abelian category is a category where "it makes sense to talk about notions such as (co)homology". The most straightforward category for this is **Ab**, the category of abelian groups. More generally one can also think about the category R-**mod**, for some ring R. Of course **Ab** can be realized as $\mathbb{Z}$-**mod**. To help motivate the abstract and categorical definitions, we will show to which familiar and algebraic notions they correspond in **Ab**. We start with the notion of a preadditive category.

9.1 Preadditive Categories

Definition 9.1. A category $\mathcal{C}$ is called *preadditive* if for all objects C, D of $\mathcal{C}$, $\mathcal{C}(C, D)$ is an abelian group and for all $f, g \in \mathcal{C}(C, D)$, $h \in \mathcal{C}(D, E)$ and $h' \in \mathcal{C}(E, C)$, where E is an object in $\mathcal{C}$, we have that $h \circ (f + g) = h \circ f + h \circ g$ and $(f + g) \circ h' = f \circ h' + g \circ h'$. Here $+$ denotes the group operation in all of the hom sets. The identities of such groups are denoted by 0 and we will also call them *zero morphisms*.

The category **Ab** is an example of a preadditive category. Given two abelian groups G and H, the hom set $\mathbf{Ab}(G, H)$ has a natural abelian group structure by adding two homomorphisms pointwise. The homomorphism that sends every element of G to the identity of H is then the identity of $\mathbf{Ab}(G, H)$. Since the collection of arrows between two objects in a preadditive category is a group, it can also be thought of as a set. So we see that a preadditive category is in particular locally small. Also note that a preadditive category with one object is nothing but a ring, similarly how we considered the one object category BG for a group G. The multiplication in this ring is the composition of arrows and so it always has a multiplicative identity, namely, the unique identity arrow.

Definition 9.2. Let $\mathcal{C}$ and $\mathcal{D}$ be preadditive categories. A functor $F \colon \mathcal{C} \to \mathcal{D}$ is called *additive* if for any two objects C and D in $\mathcal{C}$, the map $\mathcal{C}(C, D) \to \mathcal{D}(F(C), F(D)) \colon f \mapsto F(f)$ is a homomorphism of abelian groups.

Additive functors can be thought of as the "interesting" functors between preadditive categories, since they preserve the group structure of the hom sets.

Lemma 9.3. For an object C in a preadditive category $\mathcal{C}$, the Yoneda functor $\mathcal{C}(C, -) \colon \mathcal{C} \to$ **Ab** is a well-defined and additive functor.

Proof. By requiring the hom sets of $\mathcal{C}$ to have an abelian group structure, we ensure that $\mathcal{C}(C,-)$ actually takes values in **Ab**. We get additivity from the fact that function composition distributes over addition in a preadditive category. $\square$

Lemma 9.4. Let $\mathcal{C}$ be a preadditive category and A, B objects of $\mathcal{C}$. For all arrows $f \in \mathcal{C}(A,B)$, we have that $0 \circ f = 0$ and $f \circ 0 = 0$ for all relevant zero morphisms.

Proof. Suppose $f \in \mathcal{C}(A,B)$. For every zero morphism that can be precomposed with f, we have that $0 \circ f = (0+0) \circ f = 0 \circ f + 0 \circ f$, so subtracting $0 \circ f$ from both sides gives that $0 \circ f = 0$. Similarly, $f \circ 0 = 0$ for all zero morphisms that can be postcomposed with f. $\square$

Now we are ready to state and show equivalent of the Yoneda Lemma for preadditive categories.

Proposition 9.5 (Additive Yoneda Lemma)**.** Let $\mathcal{C}$ be a preadditive category, $F\colon \mathcal{C} \to \mathbf{Ab}$ an additive functor and C an object of $\mathcal{C}$. The set $\mathrm{Nat}(\mathcal{C}(C,-),F)$ has a natural abelian group structure and the Yoneda map

$$\Phi\colon\ \mathrm{Nat}(\mathcal{C}(C,-),F) \to Fc\colon \alpha \mapsto \alpha_C(\mathrm{Id}_C)$$

is an isomorphism of groups. This correspondence is also natural in C and F.

Proof. Note that since $\mathcal{C}$ is preadditive, $\mathcal{C}(C,-)$ is a functor from $\mathcal{C}$ to **Ab**. Given two natural transformations α, β from $\mathcal{C}(C,-)$ to F, define for each object A of $\mathcal{C}$ an arrow $(\alpha+\beta)_A := \alpha_A + \beta_A$. This is well-defined since **Ab** is preadditive and we will show that this defines a natural transformation from $\mathcal{C}(A,-)$ to F. Consider now two objects A, B of $\mathcal{C}$ and an arrow $f \in \mathcal{C}(A,B)$. We see, using that **Ab** is preadditive and α and β are natural, that

$$F(f)\circ(\alpha_A+\beta_A) = F(f)\circ\alpha_A + F(f)\circ\beta_A = \alpha_B\circ(f\circ -) + \beta_B\circ(f\circ -) = (\alpha_B+\beta_B)\circ(f\circ -).$$

This shows that the square

$$\begin{tikzcd}
\mathrm{Nat}(\mathcal{C}(C,A),F) \arrow[r, "\alpha_A+\beta_A"] \arrow[d, "f\circ -"'] & F(A) \arrow[d, "F(f)"] \\
\mathrm{Nat}(\mathcal{C}(C,B),F) \arrow[r, "\alpha_B+\beta_B"'] & F(B)
\end{tikzcd}$$

commutes and therefore $\alpha+\beta$ is a natural transformation from $\mathcal{C}(C,-)$ to F. We will now show that this operation makes $\mathrm{Nat}(\mathcal{C}(C,-),F)$ into an abelian group. For each object A of $\mathcal{C}$ define ω_A as the identity of the group $\mathbf{Ab}(\mathcal{C}(C,A),F(A))$. By Lemma 9.4 we see that $\omega_B \circ (f \circ -) = 0 = F(f) \circ \omega_A$ for every pair of objects A, B of $\mathcal{C}$ and arrow $f \in \mathcal{C}(A,B)$. Therefore the diagram

$$\begin{tikzcd}
\mathrm{Nat}(\mathcal{C}(C,A),F) \arrow[r, "\omega_A"] \arrow[d, "f\circ -"'] & F(A) \arrow[d, "F(f)"] \\
\mathrm{Nat}(\mathcal{C}(C,B),F) \arrow[r, "\omega_B"'] & F(B)
\end{tikzcd}$$

commutes and this shows that ω is an element of $\mathrm{Nat}(\mathcal{C}(C,-),F)$. From the definition we see that $\omega+\alpha=\alpha+\omega=\alpha$ for every $\alpha \in \mathrm{Nat}(\mathcal{C}(C,-),F)$. Next, given $\beta \in \mathrm{Nat}(\mathcal{C}(C,-),F)$, define for each object A of $\mathcal{C}$ the arrow $(-\beta)_A$ as $-\beta_A$. Note that this makes sense since β_A is an element of an abelian group and therefore has an additive inverse. For every pair of objects A, B of $\mathcal{C}$ and arrow $f \in \mathcal{C}(A,B)$, we have that $-\beta_B \circ (f\circ -) + \beta_B \circ (f \circ -) = 0 \circ (f \circ -) = 0$, so $-\beta_B \circ (f\circ -) = -(\beta_B \circ (f \circ -))$ and similarly $F(f)\circ(-\beta_A) = -(F(f)\circ \beta_A)$. The naturality of β then gives that

$$-\beta_B \circ (f \circ -) = -(\beta_B \circ (f \circ -)) = -(F(f) \circ \beta_A) = F(f) \circ (-\beta_A).$$

Therefore the diagram

$$\begin{tikzcd}
\mathrm{Nat}(\mathcal{C}(C,A),F) \arrow[r, "-\beta_A"] \arrow[d, "f\circ -"'] & F(A) \arrow[d, "F(f)"] \\
\mathrm{Nat}(\mathcal{C}(C,B),F) \arrow[r, "-\beta_B"'] & F(B)
\end{tikzcd}$$

commutes and so $-\beta$ is a natural transformation. By the definitions it follows that $\beta + -\beta = \omega$ for every $\beta \in \mathrm{Nat}(\mathcal{C}(C,-),F)$. That the operation $+$ is abelian follows directly from its definition and so we may conclude that this operation makes $\mathrm{Nat}(\mathcal{C}(C,-),F)$ into an abelian group.

Now define

$$\begin{aligned}
\Psi \colon F(C) &\to \mathrm{Nat}(\mathcal{C}(C,-),F) \\
x &\mapsto (\Psi(x)_A \colon \mathcal{C}(C,A) \to F(A) \colon f \mapsto F(f)(x) \mid A \in \mathrm{Ob}(\mathcal{C})).
\end{aligned}$$

To check that this is a well-defined map, we have to show that it takes values in $\mathrm{Nat}(\mathcal{C}(C,-),F)$. Let $x \in F(C)$, so for every object A of $\mathcal{C}$ we have to show that $\Psi(x)_A$ is a group homomorphism. So suppose that A is an object of $\mathcal{C}$ and $f, g \in \mathcal{C}(C,A)$. Then $\Psi(x)_A(f+g) = F(f+g)(x)$ and by the additivity of F and the pointwise addition of arrows in **Ab**, we get that

$$F(f+g)(x) = (F(f)+F(g))(x) = F(f)(x) + F(g)(x) = \Psi(x)_A(f) + \Psi(x)_A(g).$$

So we see that $\Psi(x)_A$ is a group homomorphism and therefore an arrow in **Ab**. The proof that $\Psi(x)$ is a natural transformation is completely analogous to the proof of this in the (covariant) Yoneda Lemma. Therefore Ψ is well-defined. Now let $\alpha, \beta \in \mathrm{Nat}(\mathcal{C}(C,-),F)$, so

$$\Phi(\alpha+\beta) = (\alpha_C + \beta_C)(\mathrm{Id}_C) = \alpha_C(\mathrm{Id}(C)) + \beta_C(\mathrm{Id}_C) = \Phi(\alpha) + \Phi(\beta),$$

and therefore Φ is a group homomorphism. The proof that Ψ is an inverse of Φ is also completely analogous to the proof of this in the (covariant) Yoneda Lemma. This shows that Φ is a group isomorphism. The proof of the naturality in C and F is again analogous to the proof of this in the (covariant) Yoneda Lemma. $\square$

Corollary 9.6 (Additive Yoneda Embedding)**.** Let $\mathcal{C}$ be a preadditive category. The functor $y\colon \mathcal{C}^{op} \to \mathbf{Ab}^{\mathcal{C}}$, which maps an object C to its representable functor $\mathcal{C}(C, -)$ and an arrow f to the natural transformation that precomposes with f in each component, is fully faithful.

Proof. The proof is completely analogous to the proof of the original Yoneda Embedding, Proposition 7.2. $\square$

9.2 Additive and Abelian Categories

Definition 9.7. An object that is both initial and terminal in a category $\mathcal{C}$ is called a *zero object*. If such a zero object exists, it is trivially unique up to unique isomorphism and we will also denote it by 0.

Definition 9.8. A category $\mathcal{C}$ is *additive* if it is preadditive, has a zero object and every two objects have a product.

In **Ab**, the trivial groups are exactly the zero objects, since there is always a unique way to define a group homomorphism to and from a trivial group.

Definition 9.9. Let $\mathcal{C}$ be an additive category, C and D objects in $\mathcal{C}$ and $f\colon C \to D$ an arrow. A *kernel of* f is an object $\ker(f)$, together with an arrow $f_k\colon \ker(f) \to C$, such that $f \circ f_k = 0$ and for any other arrow $g\colon E \to C$ such that $f \circ g = 0$, there exists an unique arrow $h\colon E \to \ker(f)$ such that the diagram

$$\begin{tikzcd}
& C \arrow[dr, "f"] & \\
& \ker(f) \arrow[u, "f_k"] \arrow[r, "0"'] & D \\
E \arrow[uur, bend left, "g"] \arrow[ur, dashed, "h"] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}$$

commutes. Similarly, a *cokernel of* f is an object $\operatorname{coker}(f)$, together with an arrow $f_c\colon D \to \operatorname{coker}(f)$ such that $f_c \circ f = 0$ and for any other arrow $g\colon D \to E$ such that $g \circ f = 0$, there exists an unique arrow $h\colon \operatorname{coker}(f) \to E$ such that the diagram

$$\begin{tikzcd}
& D \arrow[d, "f_c"] \arrow[ddr, bend left, "g"] & \\
C \arrow[ur, "f"] \arrow[r, "0"'] \arrow[drr, bend right, "0"'] & \operatorname{coker}(f) \arrow[dr, dashed, "h"] & \\
& & E
\end{tikzcd}$$

commutes.

The notion of kernels and cokernels should be familiar from algebra and our construction is a generalization of this. Given an arrow $f\colon G \to H$ in **Ab**, its kernel is the subgroup of elements of G that map to the identity of H under f. We naturally get the inclusion of the kernel into G as the associated map. We cannot directly use this construction in the categorical setting since we use that our objects are sets and that arrows are certain functions between sets. One could also define the kernel in **Ab** as being the largest subgroup whose image is only the identity of H and the categorical definition is trying to encode this. One can also think of the kernel as measuring the failure of the injectivity of f, since the kernel is trivial if and only if f is injective. This idea can be found all throughout mathematics: class groups in algebraic number theory measure "how far off a number ring is from being a principal ideal domain", (co)homology measures the failure of exactness of (co)chain complexes (we will come back to this idea), the divisor class group in algebraic geometry measures the failure of divisors to be principal .

The cokernel of f is usually defined as $H/\operatorname{im}(f)$. We naturally get a map from H to the cokernel of f by sending each element to its class in the quotient. We again cannot directly use this definition because of the same reasons as before. We also lack the notion of the image of an arrow (which will be defined in Definition 9.14) and of the quotient of objects. Another way to think of the cokernel is as the smallest quotient of H such that anything in the image of f gets mapped to zero and the categorical definition is an abstraction of this. Similar to before, the cokernel of f can be interpreted as measuring the failure of the surjectivity of f, since f is surjective if and only if its cokernel is trivial.

Lemma 9.10. In an additive category, kernels and cokernels are unique up to unique isomorphism.

Proof. From the definitions, we see that kernels (respectively cokernels) are just a specific limit (respectively colimit) and so they are unique up to unique isomorphism □

Definition 9.11. Let $\mathcal{C}$ be an additive category. We say that a monomorphism f is *normal* if it is the associated arrow of the kernel of some arrow. An epimorphism e is *normal* if it is the associated arrow of the cokernel of some arrow.

In **Ab**, monomorphisms correspond to injective homomorphisms and epimorphisms to surjective homomorphisms. It is straightforward to check that all monomorphisms are normal and the normality of epimorphisms is a consequence of the first isomorphism theorem.

Definition 9.12. A category $\mathcal{A}$ is called *abelian* if it is additive, every arrow has a kernel and cokernel and all epimorphisms and monomorphisms are normal.

Example 9.13.

- For a given ring R, the category R-**mod** is abelian. A special case of this is $R = \mathbb{Z}$, which gives **Ab**.
- If $\mathcal{C}$ is a small category and $\mathcal{A}$ abelian, then the functor category $\mathcal{A}^{\mathcal{C}}$ is again abelian [6, Example 1.6.4].
- If we choose $\mathcal{C} = \mathit{Open}(X)^{op}$, that is the opposite category of open subsets of a topological space X, and for example $\mathcal{A} = \mathbf{Ab}$, then $\mathcal{A}^{\mathcal{C}}$ consists of presheaves of abelian groups on X.

- If we choose $\mathcal{C}$ to be the poset category of natural numbers, than we can realize the category of chain complexes in $\mathcal{A}$ as a full subcategory of $\mathcal{A}^{\mathcal{C}}$. This full subcategory actually is again an abelian category [6, Example 1.6.4.2].

Definition 9.14. Let $\mathcal{A}$ be an abelian category with $M, N \in \mathrm{Obj}(\mathcal{A})$ and $f \in \mathcal{A}(M, N)$. We define the *image of* f as $\mathrm{im}(f) := \mathrm{coker}(f_k)$.

For an arrow $f\colon G \to H$ in **Ab**, the map associated to the kernel of f is just the inclusion $\ker(f) \hookrightarrow G$. This is injective so we can identify the cokernel with $G/\ker(f)$. But by the first isomorphism theorem, this is canonically isomorphic to $\mathrm{im}(f)$. We see that the categorical definition is a generalization of the notion in **Ab**.

Lemma 9.15. Let $\mathcal{A}$ be an abelian category with $M, N \in \mathrm{Obj}(\mathcal{A})$ and $f \in \mathcal{A}(M, N)$. Then f is an epimorphism if and only if for all $P \in \mathrm{Obj}(\mathcal{A})$ and $g \in \mathcal{A}(N, P)$, $g \circ f = 0$ implies $g = 0$. Similarly, f is an monomorphism if and only if for all $P \in \mathrm{Obj}(\mathcal{A})$ and $g \in \mathcal{A}(P, M)$, $f \circ g = 0$ implies $g = 0$.

Proof. Suppose that f is an epimorphism, $P \in \mathrm{Obj}(\mathcal{A})$, $g \in \mathcal{A}(N, P)$ and $g \circ f = 0$. But we also have that $0' \circ f = 0$, where $0'$ is the zero morphism from N to P, so by Lemma 9.4 it follows that $g = 0'$. Next suppose that for all $P \in \mathrm{Obj}(\mathcal{A})$ and $g \in \mathcal{A}(N, P)$, $g \circ f = 0$ implies $g = 0$. Let $g, g' \in \mathcal{A}(N, P)$ and suppose that $g \circ f = g' \circ f$, so $g \circ f - (g' \circ f) = 0$. But $g' \circ f + -g' \circ f = 0 \circ f = 0$ by Lemma 9.4, so $-(g' \circ f) = -g' \circ f$. Therefore $(g - g') \circ f = g \circ f + -g' \circ f = 0$, so by assumption $g = g'$. But this means that f is an epimorphism. The proof for the condition for f to be a monomorphism is analogous. $\square$

Lemma 9.16. Let $\mathcal{A}$ be an abelian category with $M, N \in \mathrm{Obj}(\mathcal{A})$ and $f \in \mathcal{A}(M, N)$. Then $f_c \in \mathcal{A}(N, \mathrm{coker}(f))$ is an epimorphism.

Proof. Suppose that $P \in \mathrm{Obj}(\mathcal{A})$, $g \in \mathcal{A}(\mathrm{coker}(f), P)$ and $g \circ f_c = 0$. Note that the diagram

$$\begin{tikzcd}
 & N \arrow[d, "f_c"] \arrow[ddr, bend left, "0"] & \\
M \arrow[ur, "f"] \arrow[r, "0"'] \arrow[dr, bend right, "0"'] & \mathrm{coker}(f) \arrow[dr, "g"] & \\
 & & P
\end{tikzcd}$$

commutes, so by the universal property of the cokernel this g is unique. But substituting the zero map for g still makes this diagram commute, so $g = 0$. By Lemma 9.15 we can conclude that f_c is an epimorphism. $\square$

If two arrows f, g in **Ab** are composable and we have that $g \circ f = 0$, then we know that $\mathrm{im}(f) \subseteq \ker(g)$ and so we have an inclusion $\mathrm{im}(f) \hookrightarrow \ker(g)$. This inclusion lets us talk about for example when $\mathrm{im}(f) = \ker(g)$ and the quotient group $\ker(g)/\mathrm{im}(f)$. We will now construct a map from $\mathrm{im}(f)$ to $\ker(g)$ in a general abelian category in this context. Suppose now that $\mathcal{A}$ is an abelian category, with $M, N, P \in \mathrm{Obj}(\mathcal{A})$, $f \in \mathcal{A}(M, N)$, $g \in \mathcal{A}(N, P)$ and

$g \circ f = 0$. We will construct a canonical arrow from $\operatorname{im}(f)$ to $\ker(g)$. From the definition of $\operatorname{im}(f)$, the diagram

$$\begin{tikzcd}
& M \arrow[rr, "f"] \arrow[dl, "(f_k)_c"'] & & N \\
\operatorname{im}(f) & & \ker(f) \arrow[ul, "f_k"'] \arrow[ur, "0"'] \arrow[ll, "0"] &
\end{tikzcd}$$

commutes. But now by the universal property of the cokernel, there exists a unique $u \in \mathcal{A}(\operatorname{im}(f), N)$ such that the extended diagram commutes. Then we have that $u \circ (f_k)_c = f$, so $g \circ u \circ (f_k)_c = g \circ f = 0$, by assumption. We know from Lemma 9.16 that $(f_k)_c$ is an epimorphism, so by Lemma 9.15 we know that $g \circ u = 0$. By the universal property of the kernel, there exists a unique $h \in \mathcal{A}(\operatorname{im}(f), \ker(g))$ such that the diagram

$$\begin{tikzcd}
& M \arrow[rr, "f"] \arrow[dl, "(f_k)_c"'] & & N \arrow[rr, "g"] & & P \\
\operatorname{im}(f) \arrow[urrr, dashed, "u"] \arrow[rrrr, dashed, bend right, "h"'] & & \ker(f) \arrow[ul, "f_k"'] \arrow[ur, "0"'] \arrow[ll, "0"] & & \ker(g) \arrow[ul, "g_k"] \arrow[ur, "0"'] &
\end{tikzcd}$$

commutes. This arrow h, we will call the *canonical arrow from* $\operatorname{im}(f)$ *to* $\ker(g)$.

If we consider a (co)chain complex in an abelian category, that is a sequence of arrows such that the composition of any two arrows is a zero morphism, we can consider this canonical arrow at each composition and define its cokernel as the (co)homology of the complex at that point.

Definition 9.17. Let $\mathcal{A}$ be an abelian category, with $M, N, P \in \operatorname{Obj}(\mathcal{A})$, $f \in \mathcal{A}(M, N)$, $g \in \mathcal{A}(N, P)$ and $g \circ f = 0$. We say that the sequence $M \xrightarrow{f} N \xrightarrow{g} P$ is *exact at* N, if $g \circ f = 0$ and the canonical arrow from $\operatorname{im}(f)$ to $\ker(g)$ is an isomorphism. We call a sequence of morphisms in $\mathcal{A}$ *exact* if it is exact at every object. Exact sequences of the form $0 \to M \xrightarrow{f} N \xrightarrow{g} P \to 0$ are called *short exact sequences.*

In Definition 9.17 we did not give names to the arrows to and from the zero object, since by definition they can only be zero morphisms. We will continue with this convention, also when such maps occur in diagrams. We consider a short exact sequence $0 \to M \xrightarrow{f} N \xrightarrow{g} P \to 0$ in **Ab**. Exactness at P means that $\operatorname{im}(g) = \ker(P \to 0) = P$, so g is surjective. By the first isomorphism theorem we get that $P \cong N/\ker(g)$, but exactness at N means that $\ker g = \operatorname{im} f$, so $P \cong N/\operatorname{im}(f)$. Exactness at M gives that $\ker(f) = \operatorname{im}(0 \to M) = 0$, so f is injective. Therefore we can identify $\operatorname{im}(f)$ with M and, with some abuse of notation, we can write $P \cong N/M$. So intuitively we should think about short exact sequences as expressing one object as the quotient of two others.

Definition 9.18. Let $\mathcal{A}, \mathcal{B}$ be abelian categories and $F\colon \mathcal{A} \to \mathcal{B}$ an additive functor. We say that F is *left exact* if whenever $0 \to M \xrightarrow{f} N \xrightarrow{g} P \to 0$ is a short exact sequence in $\mathcal{A}$, the sequence $0 \to F(M) \xrightarrow{F(f)} F(N) \xrightarrow{F(g)} F(P)$ is an exact sequence in $\mathcal{B}$. Similarly, we say that F is *right exact* if whenever $0 \to M \xrightarrow{f} N \xrightarrow{g} P \to 0$ is a short exact sequence in $\mathcal{A}$, the sequence $F(M) \xrightarrow{F(f)} F(N) \xrightarrow{F(g)} F(P) \to 0$ is an exact sequence in $\mathcal{B}$. A functor that is both left and right exact, is called *exact*.

An example of a right exact functor is, given a commutative ring R and a R-module M, the functor $M \otimes_R -\colon R\text{-}\mathbf{mod} \to R\text{-}\mathbf{mod}$. This functor assigns to the R-modules N and P the R-modules $M \otimes_R N$ and $M \otimes_R P$ respectively. An arrow $f \in R\text{-}\mathbf{mod}(N, P)$ gets mapped to the function $M \otimes_R N \to M \otimes_R P\colon m \otimes n \mapsto m \otimes f(n)$ (which is extended linearly). The proof of the right-exactness of this functor (and the fact that it is well-defined) can be found in [7, Proposition 2.18]. We will now show that (covariant) representable functors are left exact.

Proposition 9.19. Let $\mathcal{A}$ be an abelian category and $A \in \mathrm{Obj}(\mathcal{A})$. Then the representable functor $\mathcal{A}(A, -)\colon \mathcal{A} \to \mathbf{Ab}$ is left exact.

Proof. We know from Lemma 9.3 that $\mathcal{A}(A, -)$ is additive. Suppose that $0 \to M \xrightarrow{f} N \xrightarrow{g} P \to 0$ is a short exact sequence in $\mathcal{A}$. We will begin by proving that the exactness of the sequence at M implies that f is a monomorphism. From the universal property of the kernel it follows that 0, of course together with the map $0 \to 0$, is a kernel of $0 \to m$. Then from the definition of the cokernel, it follows that 0 is a cokernel of $0 \to 0$. But this is then exactly $\mathrm{im}(0 \to m)$. Since our sequence is exact at m, we have that $\ker(f)$ is isomorphic to $\mathrm{im}(0 \to m)$ and therefore a zero object. In particular, we have that $f_k = 0$. Now suppose that $M' \in \mathrm{Obj}(\mathcal{A})$ and $h \in \mathcal{A}(M', M)$ such that $f \circ h = 0$. But since 0 is a kernel of f, there exists a unique $i \in \mathcal{A}(M', 0)$ (the zero morphism of course) such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[rr, "f"] & & N \\
& & 0 \arrow[ul] \arrow[ur] & \\
& & M' \arrow[u, "i"] \arrow[uul, bend left, "h"] \arrow[uur, bend right, "0"'] &
\end{tikzcd}$$

commutes. But then $h = 0 \circ i = 0$, so f is a monomorphism by Lemma 9.15. Now suppose that $\varphi, \psi \in \mathcal{A}(A, M)$ such that $f \circ \varphi = f \circ \psi$. Since f is a monomorphism it follows that $\varphi = \psi$ and so the morphism of abelian groups $f \circ -\colon \mathcal{A}(A, M) \to \mathcal{A}(A, N)$ is injective. But this means that the sequence $0 \to \mathcal{A}(A, M) \xrightarrow{f\circ -} \mathcal{A}(A, N) \xrightarrow{g\circ -} \mathcal{A}(A, P)$ in $\mathbf{Ab}$ is exact at $\mathcal{A}(A, M)$.

Now it only rests to show that the sequence $0 \to \mathcal{A}(A, M) \xrightarrow{f\circ -} \mathcal{A}(A, N) \xrightarrow{g\circ -} \mathcal{A}(A, P)$ is exact at $\mathcal{A}(A, N)$. Since we are now working in $\mathbf{Ab}$, this is equivalent to showing that for all $\varphi \in \mathcal{A}(A, N)$, $g \circ \varphi = 0$ if and only if there exists $\psi \in \mathcal{A}(A, M)$ such that $\varphi = f \circ \psi$. Now let

$\varphi \in \mathcal{A}(A, N)$ and first suppose that there exists $\psi \in \mathcal{A}(A, M)$ such that $\varphi = f \circ \psi$. By the exactness of $M \xrightarrow{f} N \xrightarrow{g} P$ at N, we have that $g \circ f = 0$. Then $g \circ \varphi = g \circ f \circ \psi = 0 \circ \psi = 0$. Now suppose that $g \circ \varphi = 0$. We have already established that 0 is a kernel of f. Suppose that $C \in \mathrm{Obj}(\mathcal{A})$ and $j \in \mathcal{A}(M, C)$ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "f"] \arrow[d, "\mathrm{Id}_M"'] \arrow[ddl, bend right, "j"'] & N \\
 & M & 0 \arrow[l] \arrow[u] \arrow[ul] \arrow[dll, bend left] \\
C & &
\end{tikzcd}$$

commutes. Then j is certainly the unique arrow that can be added from the lower copy of M to C such that the new diagram commutes. This shows that M, together with the arrow Id_M, is a cokernel of f_k and therefore the image of f. By exactness of $M \xrightarrow{f} N \xrightarrow{g} P$ at N, the canonical arrow h from $M = \mathrm{im}(f)$ to $\ker(g)$ exists and is an isomorphism. Therefore the diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "f"] \arrow[drr, bend right, "h"'] & N \arrow[rr, "g"] & & P \\
 & & & \ker(g) \arrow[ul, "g_k"] \arrow[ur, bend right, "0"'] &
\end{tikzcd}$$

commutes. Since we assumed that $g \circ \varphi = 0$, we can use the universal property of the kernel to see that there exists a unique $r \in \mathcal{A}(A, \ker(g))$ such that the extended diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "f"] \arrow[ddrr, bend right, "h"'] & N \arrow[rr, "g"] & & P \\
 & & & A \arrow[ul, "\varphi"'] \arrow[ur, "0"] \arrow[d, dashed, "r"'] & \\
 & & & \ker(g) \arrow[uul, bend left, "g_k"] \arrow[uur, bend right, "0"'] &
\end{tikzcd}$$

commutes. But now we see that $\varphi = f \circ h^{-1} \circ r$ and $h^{-1} \circ r \in \mathcal{A}(A, M)$ and this is exactly what we needed to show. $\square$

Now we are ready to state and provide a sketch of the proof of *Mitchell's embedding theorem.* For a more complete exposition, we refer to [8, Part I].

Theorem 9.20 (Mitchell's Embedding Theorem)**.** Let $\mathcal{A}$ be a small abelian category. Then there exists a ring R, with multiplicative identity, and an exact and fully faithful functor from $\mathcal{A}$ to R-**mod**.

Proof. We begin by considering the functor $y \colon \mathcal{A}^{op} \to \mathbf{Ab}^{\mathcal{A}}$ from Corollary 9.6. We know from Corollary 9.6 that this functor is fully faithful. By Proposition 9.19, we know that all of these representable functors lie in the full subcategory of left exact functors, which we will denote by $\mathcal{L}$. The category $\mathcal{L}$ turns out to be abelian and the functor $y\colon \mathcal{A}^{op} \to \mathcal{L}$ is also exact. Next one shows that $\mathcal{L}$ has an *injective cogenerator* I. We define $R = \mathcal{L}(I, I)$, where composition of arrows is the multiplication of the ring. Note that Id_R is the multiplicative identity of R. For a given object $A \in \mathcal{L}$, $\mathcal{L}(A, I)$ is naturally a left R-module, since elements of R act on this abelian group by composition. This defines a functor $\mathcal{L}(-, I)\colon \mathcal{L}^{op} \to R$-**mod**. From the properties of I it can be shown that this functor is once again exact and fully faithful. The composition $\mathcal{L}(-, I) \circ y\colon \mathcal{A} \to R$-**mod** is then a (covariant) exact and fully faithful functor. $\square$

References

[1] Emily Riehl. *Category Theory in Context.* Dover Publications, 2016.

[2] nLab authors. *large category.* `https://ncatlab.org/nlab/show/large+category`. Revision 25. Jan. 2024.

[3] Terrence Tao. *Yoneda's lemma as an identification of form and function: the case study of polynomials.* 2023. URL: `https://terrytao.wordpress.com/2023/08/25/yonedas-lemma-as-an-identification-of-form-and-function-the-case-study-of-polynomials/` (visited on 01/20/2024).

[4] Tom Leinster. *Basic Category Theory.* 2016. arXiv: `1612.09375 [math.CT]`.

[5] nLab authors. *simplicial set.* `https://ncatlab.org/nlab/show/simplicial+set`. Revision 88. Jan. 2024.

[6] Charles A. Weibel. *An introduction to homological algebra.* Vol. 38. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1994, pp. xiv+450. ISBN: 0-521-43500-5; 0-521-55987-1. DOI: `10.1017/CBO9781139644136`. URL: `https://doi.org/10.1017/CBO9781139644136`.

[7] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra.* Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969, pp. ix+128.

[8] R. G. Swan. *Algebraic K-theory.* Lecture Notes in Mathematics, No. 76. Springer-Verlag, Berlin-New York, 1968, pp. iv+262.